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RECURSIVE COMPETITIVE EQUILIBRIUM
A Parametric Example

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In this paper we demonstrate how the neoclassical model of stochastic growth can be decentralized in a recursive competitive framework. We provide a closed form parametric example to facilitate a conceptual understanding of the more abstract variations of this line of research.

1. Introduction

In recent years considerable interest has been focused on the integration of stochastic growth theory and modern financial theory, with particular reference to the pricing of capital assets. The analytic foundation of this work in its simplest version, is the observation by Brock (1979,1982) and by Prescott and Mehra (1980) that the investment and consumption policy functions arising as a solution to the *central planning* stochastic growth problem

$$\text{(P)} \qquad \max \mathrm{E}\left\{ \sum_{t=0}^{\infty} \beta u(c_t) \right\}, \quad \text{subject to}$$

$$c_t + k_{t+1} \leqq \lambda_t f(k_t), \quad k_0, \lambda_0 \text{ given},$$

may be regarded as the aggregate investment and consumption functions arising from a *decentralized* homogeneous consumer economy.

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Problem (P) has the usual interpretation: [1] preferences are time separable with period discount factor β and period utility function $u(\ )$ defined over period consumption c_t; k_t denotes capital available for production in period t while $f(\ )$ represents the period technology which is shocked by the stochastic factor λ_t. The shock sequence $\{\lambda_t\}, \underline{\lambda} \leqq \lambda_t \leqq \bar{\lambda}$ follows a Markov process with period distribution function $H(\cdot)$; that is

$$H(\lambda^*_{t+1}) = \int_{\underline{\lambda}}^{\lambda^*_{t+1}} \int_{\underline{\lambda}}^{\bar{\lambda}} F(\mathrm{d}\lambda_{t+1}, \lambda_t) H(\mathrm{d}\lambda_t).$$

The planner thus chooses consumption and investment policies which maximize the expected present value of discounted utility. Both Brock (1979,1982) and Prescott and Mehra (1980) present an alternative interpretation of this centrally planned economy which is consistent with competitive equilibrium. Their techniques, however, vary substantially. Brock uses variational methods while Prescott and Mehra use recursive equilibrium theory.

The intent of this paper is to present a closed form parametric example of the recursive competitive equilibrium (RCE) theory developed in Prescott and Mehra (1980), with a view to providing a cogent, intuitive understanding of this contemporary and versatile modelling technique. [2]

The reader is referred to Prescott and Mehra (1980) for a definition of a recursive competitive equilibrium. That paper also establishes the optimality of recursive equilibria and the supportability of a Pareto optimum by a recursive equilibrium. [3]

Very briefly, a RCE is characterized by *time invariant* functions of a limited number of 'state variables' which summarize the effects of past decisions and current information. These functions (decision rules) include (a) a pricing function, (b) a value function, (c) a period allocation policy specifying the individual's decision, (d) period allocation policies specifying the decision of each firm and (e) a function specifying the law of motion of the capital stock.

[1] Problem (P) is studied in Brock and Mirman (1972), Mirman and Zilcha (1975), Mirrlees (1974) and others when λ is i.i.d. Donaldson and Mehra (1983) consider the case when the shocks are correlated.

[2] Examples of other models that can be cast in a RCE framework can be found in Prescott and Mehra (1980, sect. 8).

[3] The framework in that paper is considerably more general, in particular multiple capital and consumption goods are allowed.

2. An example

In this section we present a *closed form* solution of a RCE. We consider a special case of the economy considered in Prescott and Mehra (1980). There is one capital good and this is assumed to produce two goods – a consumer good and an investment (capital) good. At the beginning of each period, firms observe the shock to productivity (λ_t) and purchase capital and labor from individuals at competitively determined rates. Both capital and labor are used to produce the two output goods. Individuals use their proceeds from the sale of capital to buy the consumption good (c_t) and the investment good (i_t) at the end of the period. This investment good is used as capital (k_{t+1}) available for sale to the firm next period. This process continues recursively. For a further elaboration and an application to multiperiod asset pricing, see Donaldson and Mehra (1984). [4]

For the purpose of this example we restrict preferences to be logarithmic and assume Cobb–Douglas technology. Specifically,

$$U(c) = \ln c, \qquad f(k, l) = k^{\alpha} l^{1-\alpha},$$

and λ is independently and identically distributed with bounded support.

These conditions are sufficient to ensure a closed form solution for a RCE. It is well known (and can be readily demonstrated) that the optimal stationary consumption and investment policies for the central planning problem

$$w(k_0, \lambda_0) = \max \mathrm{E}\left\{ \sum_{t=0}^{\infty} \beta^t \ln c_t \right\}, \quad \text{such that}$$

$$c_t + i_t \leqq \lambda_t k_t^{\alpha} l_t^{1-\alpha}, \quad \lambda_0, k_0 \text{ given}, \quad l_t = 1 \ \forall t \quad \text{and} \quad \alpha, \beta < 1,$$

are

$$c_t = (1 - \alpha\beta)\lambda_t k_t^{\alpha} \quad \text{and} \quad i_t \equiv k_{t+1} = \alpha\beta\lambda_t k_t^{\alpha}.$$

i_t, the investment good held at time t, is used as capital k_{t+1} at time $t+1$. Further $w(k_0, \lambda_0) = A + B \ln k_0 + C \ln \lambda_0$ where A, B and C are constants.

[4] See also Mehra and Prescott (1984).

To cast this problem as a recursive competitive equilibrium we introduce some additional notation. Let k_t denote the capital holdings of a particular individual at time t and $\underline{k}_t$ the distribution of capital amongst other individuals in the economy. For a discussion as to why this distinction is necessary (and any further clarifications), see Prescott and Mehra (1980) especially pp. 1369–1370. Clearly in equilibrium $k_t = \underline{k}_t$ for our homogeneous consumer economy. In addition let P_{i}, P_{c} and P_{k} be the price of the investment, consumption and capital goods respectively and P_{l} be the wage rate. The 'state variables' characterizing the economy are $(\underline{k}, \lambda)$ and the individual $(k, \underline{k}, \lambda)$.

In the decentralized version of this economy the problem facing a typical household is

$$\max \mathrm{E}\left\{ \sum_{t=0}^{\infty} \beta^t \ln c_t \right\}, \quad \text{subject to}$$

$$P_{\mathrm{c}}(\underline{k}_t, \lambda_t) c_t + P_{\mathrm{i}}(\underline{k}_t, \lambda_t) i_t \leqq P_{\mathrm{k}}(\underline{k}_t, \lambda_t) k_t + P_{\mathrm{l}}(\underline{k}_t, \lambda_t) l_t,$$

$$k_{t+1} = i(k_t, \underline{k}_t, \lambda_t), \qquad l_t \leqq 1, \qquad \underline{k}_{t+1} = i(\underline{k}_t, \underline{k}_t \lambda_t).$$

With capital and labor priced competitively each period, the firm's objective function is especially simple – maximize period profits. The firm's problem then is

$$\max\{ P_{\mathrm{c}}(\underline{k}_t, \lambda_t) c_t + P_{\mathrm{i}}(\underline{k}_t, \lambda_t) i_t - P_{\mathrm{k}}(k_t, \lambda_t) k_t - P_{\mathrm{l}}(k_t, \lambda_t) l_t \},$$

subject to

$$c_t + i_t \leqq \lambda_t k_t^{\alpha} l_t^{1-\alpha}.$$

We use the symbols (c, i, k, l) to characterize the commodity points for the firm and the consumer. To clarify, the c in the commodity point of the firm is a function specifying the consumption good supplied by the firm; perhaps a more correct notation would be $c^{\mathrm{s}}(\underline{k}, \lambda)$. Similarly, the c in the commodity point of the individual is the amount of the consumption good demanded by the individual and should technically be written as $c^{\mathrm{d}}(k, \underline{k}, \lambda)$. In equilibrium, [5] of course $c^{\mathrm{s}} = c^{\mathrm{d}}$. Since there is little room for confusion we stick to the simplified notation of this paper.

[5] As mentioned earlier, in equilibrium $k_t = \underline{k}_t$.

Under these conditions it can be shown that the following functions satisfy the definition of a RCE:

(a) Value function

$$v(k_0, \underline{k}_0, \lambda_0) = \mathrm{E}\left[\sum_{t=0}^{\infty} \beta^t \ln c_t\right].$$

It is readily demonstrated that $v(\underline{k}, \underline{k}, \lambda) = w(\underline{k}, \lambda)$.

(b) Pricing functions

$$P_{\mathrm{c}}(\underline{k}_t, \lambda_t) = P_{\mathrm{i}}(\underline{k}_t, \lambda_t) = 1.$$

(We have chosen the consumption good to be the numeraire.)

$$P_{\mathrm{k}}(\underline{k}_t, \lambda_t) = \alpha\lambda_t \underline{k}_t^{\alpha-1}, \qquad P_{\mathrm{l}}(\underline{k}_t, \lambda_t) = (1-\alpha)\lambda_t \underline{k}_t^{\alpha}.$$

(c) Consumption and investment functions for the individual

$$c_t \equiv c(k, \underline{k}, \lambda_t) = (1-\alpha\beta)\lambda_t \underline{k}_t^{\alpha-1}\{\alpha(k_t - \underline{k}_t) + \underline{k}_t\},$$

$$l_t \equiv l(k, \underline{k}, \lambda_t) = 1,$$

$$i_t \equiv i(k, \underline{k}, \lambda_t) = \alpha\beta\lambda_t \underline{k}_t^{\alpha-1}\{\alpha(k_t - \underline{k}_t) + \underline{k}_t\},$$

$$k_t \equiv i_{t-1}.$$

(d) Decision rules for the firm

$$c_t \equiv c(\underline{k}_t, \lambda_t) = (1-\alpha\beta)\lambda_t \underline{k}_t^{\alpha},$$

$$l_t \equiv l(\underline{k}_t, \lambda_t) = 1,$$

$$i_t \equiv i(\underline{k}_t, \lambda_t) = \alpha\beta\lambda_t \underline{k}_t^{\alpha},$$

$$k_t \equiv i_{t-1}.$$

(e) Law of motion for the capital stock

$$\underline{k}_{t+1} = i(\underline{k}_t, \underline{k}_t, \lambda_t) = \alpha\beta\lambda_t \underline{k}_t^{\alpha}.$$

We have omitted the details of the derivations. The general technique is illustrated in Donaldson and Mehra (1984). We conjecture that this is the only closed form representation of an RCE.

Having formulated expressions for the prices of the various assets and their laws of motion, it is a relatively simple matter to calculate rates of returns (price ratios) and study their dynamics. For an illustration of an application to risk premia, see Donaldson and Mehra (1984).

Other topics amenable to analysis include the term structure of interest rates, and the dynamics of multiperiod pricing.

3. Concluding remarks

In this paper we demonstrate how the neo-classical model of stochastic growth on which much of our economic intuition is predicated can be decentralized and cast in a recursive competitive framework. We provide a concrete closed form example to facilitate a conceptual understanding of the more abstract variations of this line of research and suggest some applications to financial theory.

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