ON THE TERM STRUCTURE OF INTEREST RATES*

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This paper tests the one good stochastic growth model with respect to its ability to explain the term structure of real interest rates. We undertake both a qualitative and quantitative analysis. First we assess the changing shape of the yield curve over the model economy's 'business cycle' and compare our results with what is empirically observed. Second, we employ the model to study various implications of informational and allocative efficiency, properties which the artificial economy must possess. It is found, for example, that long-term rates are less volatile than short-term rates and that holding premia can be highly correlated over time. Third, we study the time-varying risk premium implicit in the economy's forward rate structure. A purely quantitative assessment of the model's explanatory power is also provided.

1. Introduction

In a recent paper, Mehra and Prescott (1985) test a class of infinite-horizon representative-agent stochastic growth models with respect to their ability to

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replicate, quantitatively, observed market risk premia. They demonstrate that this class of models fails this important test, a result that has stimulated considerable professional discussion.

More significant, perhaps, is the innovative methodology introduced in the study. It has two essential characteristics:

(i) **Cross-model verification.** The model's parameter values must be determined by micro- or macroeconomic studies which are conducted independently of the phenomenon being studied.

(ii) **Internal consistency.** The 'market' and risk-free rates are determined entirely endogenously within the model, thereby constraining them to be consistent with one another.2

This paper extends this methodology to the study of the term structure of interest rates. Our specific context for this exercise is the one-good neoclassical stochastic growth model. This choice can be justified in a number of ways: (1) While by no means the most sophisticated construct available, it is, nevertheless, the theoretical underpinning of much dynamic modelling (e.g., real business cycle models) and thus constitutes an appropriate starting point for our comparative analysis. Furthermore, its 'stripped-down' simplicity is likely to afford the most direct enrichment of intuition. (2) For the neoclassical model we can readily characterize the Markov processes on the real state variables (consumption, investment, and output) and compute their stationary distributions. This, in turn, allows us to derive the exact probability distribution of prices of bonds of various maturities. From this data, we can calculate the various rates of return, forward premia, and price and rate autocorrelations critical to our study. (3) In our consideration of bond market informational and allocative efficiencies, we intend to compare our results with those obtained from a variety of econometric studies, where stationarity of the relevant time series is typically assumed. We therefore focus on the model's stationary equilibria. Prescott and Mehra (1980) have shown, for this case, that the optimal stationary allocations arising from this model may be regarded as the competitive equilibrium of a homogeneous consumer econ-

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1Mehra and Prescott (1985) actually consider an exchange economy with output growing at an exogenously determined rate. Mehra (1987) demonstrates that modifying the technology in an exchange economy to admit production and capital accumulation will not increase the equity premium.

2For example, if two technologies are specified independently, one risk-free and one risky, it will not necessarily be the case that the estimates so derived would be mutually consistent.

3This model has been studied in Brock and Mirman (1972, 1973), Levhari and Srinivasan (1969), Mirman and Zilcha (1975), Mirrlees (1974), and others when the shocks to technology are i.i.d. Donaldson and Mehra (1983) consider the case when the shocks are correlated.
economy in recursive equilibrium. Thus, we are assured that our equilibrium security prices are perfectly consistent both with the real and financial aspects of the economy.

The term structure literature is so extensive that it would be impossible to evaluate our model in the context of all its strands. We therefore restrict our attention to three specific areas. First we study the behavior of the real term structure over the artificial economy's 'business cycle' and assemble the available empirical evidence in the area. This is followed by an evaluation of the model's quantitative output and its degree of conformity with observed real interest rate levels. Second, we employ the model to study various implications of informational and allocative efficiency − properties our artificial economy must, by assumption, possess. We find, for example, that long-term rates are less volatile than short-term rates (a fact supported by the empirical literature) and that holding premia can be highly correlated over time. Third, we evaluate the accuracy of forward rates as estimators of future spot rates − a traditional issue in the term structure literature − in our model context. Our model is found to exhibit time-varying forward premia, a fact also consistent with the empirical data.

To facilitate quantitative comparisons we solve the neoclassical model explicitly. This means the undertaking of an exhaustive numerical analysis for a wide range of parameter values − not only those values consistent with macro- and microeconomic studies but also a broad range of additional values in order that we may properly evaluate model sensitivity. Theoretic propositions supplement these numerical results in order to enhance our understanding of their underlying intuition.

An outline of the paper is as follows: Section 2 reviews the basic model and derives expressions for the pricing of discount bonds of various maturities. In section 3 (and the appendix) we present an outline of the numerical procedures employed. Section 4 details the magnitude and behavior of the yield curve over the cycle. Section 5 considers the relative volatility of long

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Footnotes:

4For a full equivalence, even in the case of nonstationary equilibria, a certain transversality condition at infinity must also be satisfied. This issue is fully detailed in works by Araujo and Scheinkman (1983), Benveniste and Scheinkman (1982), and Weitzman (1973).

In addition to Prescott and Mehra (1980), Brock (1982) has developed a general equilibrium model with production that characterizes equilibrium in financial markets. Other related work includes the continuous time model of Cox, Ingersoll, and Ross (1985).

5Few topics in financial economics have as long been the subject of continuous research interest as the term structure of interest rates. Indeed, this literature goes back nearly a century and is too extensive to be fully detailed here. Fisher (1896), Hicks (1946), and Meiselman (1962) are some of the more prominent earlier researchers. More recent theoretical studies include the works of Cox, Ingersoll, and Ross (1981, 1985), LeRoy (1982, 1984a,b), Richard (1978), Singleton (1980), and Shiller (1979). LeRoy (1982, 1984a,b), Singleton (1980), and Shiller (1979) utilize discrete time methodology, while Richard (1978) and Cox et al. (1981, 1985) conduct their analysis in continuous time. Much of this work evaluates various forms of the unbiased expectations hypothesis.
and short rates, while section 6 explores the correlation through time of successive interest rate changes and holding period returns. Section 7 studies the information content of the yield curve and, in particular, focuses on the issue of the time-varying forward premium. Section 8 concludes the paper.

2. The economy

2.1. Model overview

Consider the central planning dynamic stochastic growth problem

\[
\max_{(c_t, k_{t+1})} \mathbb{E} \left( \sum_{t=0}^{\infty} \beta^t u(c_t) \right), \tag{P}
\]

subject to

\[c_t + k_{t+1} \leq f(k_t) \lambda_t, \quad k_0 \text{ given},\]

in which the representative agent's preferences are assumed time-separable with period discount factor \(\beta\) and utility function \(u(\cdot)\) defined over period consumption \(c_t\). In this setting, it is customary to denote capital available for production in period \(t\) by \(k_t\); \(f(\cdot)\) thus represents the period technology which is subjected to the multiplicative stochastic factor \(\lambda_t\) and \(\mathbb{E}\) denotes the expectations operator. The shock sequence \(\{\lambda_t\}, 0 < \lambda_0 \leq \lambda_t \leq \lambda < \infty\), is assumed to follow a Markov process with transition density \(F(d\lambda_{t+1} | \lambda_t)\) and invariant measure \(G(d\lambda)\).

Problem (P) has been studied by Brock and Mirman (1972, 1973) and Donaldson and Mehra (1983), among others (see footnote 3). Under appropriate assumptions the basic results are two-fold: (1) optimal, time-invariant consumption \(c(k, \lambda)\) and saving \(s(k, \lambda)\) policy functions exist which solve (P), and (2) by the repeated application of these policies, the economy converges to a well-defined steady state.\(^6\) Another property of this model that we shall subsequently employ is that each policy function is increasing as a function of both state variables.

Turning now to the pricing of bonds, we introduce an implicit financial instruments market where a riskless asset is traded, this asset being in zero net supply. Its price is obtained in the usual way from the first-order conditions of the representative consumer. In particular, given the current state \((k_t, \lambda_t)\), the equilibrium price \(P = P(k_t, \lambda_t)\) of a one-period 'pure discount bond' (i.e., a security which unconditionally pays one unit of

\(^6\)Precise sufficient conditions under which properties (1) and (2) hold can be found in, e.g., Donaldson and Mehra (1983) or Futia (1982).
consumption at time $t + 1$) is therefore

$$P(k_t, \lambda_t) = \beta \int \frac{u'(c(k_{t+1}, \lambda_{t+1}))}{u'(c(k_t, \lambda_t))} F(d\lambda_{t+1}; \lambda_t).$$

(1)

By analogy, an $n$-period pure discount bond would represent an unconditional promise to pay one unit of the consumption good $n$ periods hence. To price $n$-period pure discount bonds, it is necessary to consider the $n$-period transition function on the state variables. Denote the joint capital stock–shock conditional one-period distribution function by

$$H_t(k_{t+1}, \lambda_{t+1} | k_t, \lambda_t).$$

We can thus recursively define the $n$-period conditional distribution function by

$$H_n(k_{t+n}, \lambda_{t+n} | k_t, \lambda_t) = \int H_t(k_{t+n}, \lambda_{t+n} | k_{t+n-1}, \lambda_{t+n-1})$$

$$\times H_{n-1}(dk_{t+n-1}, dk_{t+n-1} | k_t, \lambda_t), \quad n \geq 2.$$  

(2)

The price of an $n$-period discount bond maturing in period $t + n$ can now be formulated:

$$P_n(k_t, \lambda_t) = \beta^n \int \frac{u'(c(k_{t+n}, \lambda_{t+n}))}{u'(c(k_t, \lambda_t))} H_n(dk_{t+n}, d\lambda_{t+n} | k_t, \lambda_t).$$

(3)

Given these prices, we have all the information necessary to calculate the term structure of real rates. The model allows a multiplicity of consistent definitions of the term structure, two of which we will consider:

(i) **The Conditional Term Structure**: Given state $(k_t, \lambda_t)$, the conditional term structure $(r_n(k_t, \lambda_t))$ satisfies:

$$r_n(k_t, \lambda_t) = \left[ \frac{1}{P_n(k_t, \lambda_t)} \right]^{1/n} - 1.$$  

(4)

(ii) **The Average Term Structure**: The unconditional or average term structure $(r_n)$ would be computed as the average of the conditional rates:

$$r_n = \int \left[ \frac{1}{P_n(k_t, \lambda_t)} \right]^{1/n} H(dk_t, d\lambda_t) - 1.$$  

(5)

Any default-free coupon bearing bond may be considered to be a portfolio of pure discount bonds. Our analysis can therefore be applied to price these bonds as well.
In what follows we examine such issues as the changing yield curve and the predictve power of forward rates. These quantities can be calculated explicitly, given a knowledge of the probabilistic evolution of the economy. We do this partially in the context of a numerical simulation, an overview of which is the subject of our next section.

3. Numerical simulation

For this exercise we explicitly solved problem (P) and computed the $n$-step transition functions $\{H_n(\text{dk}_{t+n}, \text{d}\lambda_{t+n} | k_t, \lambda_t)\}$, and the corresponding term structure of interest rates for the case of power preferences $u(c) = (c^\gamma - 1)/\gamma$, and Cobb–Douglas technology $f(k) = \frac{\beta}{\alpha}k^\alpha \lambda$, $\alpha \in \{0.25, 0.36, 0.5\}$. The parameter $\gamma$ was chosen from the set $\{-2, -1, 0, 0.5, 0.33, 1\}$, where $\gamma = 0$ corresponds to logarithmic utility, while $\beta$ could assume values chosen from $\{0.8, 0.9, 0.95, 0.96, 0.99\}$. The shock to technology $\lambda$ was governed by a three-state Markov process with transition function $f(\lambda_{t+1}; \lambda_t)$, as described by the transition probability matrix

$$
\begin{align*}
\lambda_{t+1} &= 1 - \Delta \\
\lambda_{t+1} &= 1 + \Delta \\
\lambda_t &= 1 - \Delta \begin{bmatrix} \pi & (1-\pi)/2 & (1-\pi)/2 \\ (1-\pi)/2 & \pi & (1-\pi)/2 \\ (1-\pi)/2 & (1-\pi)/2 & \pi \end{bmatrix},
\end{align*}
$$

where $\pi \in \{0.333, 0.54, 0.7, 0.9\}$ and $\Delta \in \{0.03, 0.1, 0.5\}$.\footnote{The wide range of parameter values was chosen in order to allow a sensitivity analysis of our qualitative and quantitative results. Included in this set are those specific values chosen by Hansen (1985) and Kydland and Prescott (1982).}

Every choice of parameters $\alpha$, $\gamma$, $\beta$, and $\pi$ generates a range of possible values for the capital stock variable $k$. Our example was chosen so that for all parameter choices these values were constrained within the unit interval $[0, 1]$.\footnote{The coefficient preceding the production technology can be adjusted to govern the range of the stationary capital stock distribution. The choice of $2/3$ guarantees that output and thus capital stock cannot exceed one.}

We utilized an equal length partition $\Gamma$ of this interval with unit length 0.01.\footnote{Our program allowed the norm of the partitions to be a choice variable. Using a partition with interval unit length 0.001 did not significantly alter the results.}

In order to derive the term structure, it was first necessary to compute the optimal savings and consumption policies for (P). Here we followed the customary dynamic programming method of seeking a fixed point to the related functional equation by a sequence of approximating iterations. Using these policy functions, we next generated the time series corresponding to
the approximating stationary distribution. From this information the term structure can be calculated. This procedure is detailed in the appendix.

In what follows we supplement theoretical results with intuition gained from the simulation just described. The model’s characteristics are then contrasted with the corresponding properties of the observed yield curve as drawn from the empirical literature.

Many of the studies we will cite deal with nominal rates while our model focuses exclusively on the real term structure. We feel these comparisons are relevant for two reasons. First, Mishkin (1987) demonstrates that real and nominal 12-month term structures move in tandem, so that fluctuations in one are closely mirrored by fluctuations in the other. This is consistent with earlier work by Litterman (1980) who demonstrated the orthogonality of nominal short rates and inflation rates. Second, nominal and real rates frequently exhibit very similar qualitative properties in intertemporal monetary models. See, for example, the model structure in Danthine et al. (1987) and Backus et al. (1989).

Bearing this caveat in mind, we first turn to a qualitative description of the behavior of the real yield curve.

4. The yield curve: Behavior over the cycle and asymptotic properties

There are two basic topics we propose to consider in this section.

4.1. Cyclical behavior

The principle issue here is the behavior of the term structure over the ‘business cycle’ and the empirical literature is fairly unambiguous in this regard. Stambaugh (1988) conclusively demonstrates that upward-sloping nominal term structures precede recoveries while downward-sloping or ‘inverted’ term structures precede recessions. This is consistent with the commonly held view that interest rates are procyclical. See also the classic Kessel (1965) reference.

We are aware of only one empirical study which considers the behavior of real interest rates over the cycle, that of Prescott et al. (1983). These authors detect a negative lagged relationship between real interest rates and output levels. The contemporaneous relationship between output and real rates is also found to be negative, though not strongly significant.

Turning next to an assessment of the model’s performance, we note that the effect of the random shock to technology, coupled with the fact that the capital stock is, in any period, the prior period’s savings, is to generate persistence (cyclical behavior) in the time series of capital stock, consumption, and output. We find that the shape of the yield curve changes dramati-
tally depending on the state of the economy relative to this cycle. These changes are summarized in the following propositions and numerical observations (properties which hold for all combinations of parameter values). Here we identify the top and bottom of the cycle with, respectively, the maximum and minimum stationary capital stock–shock combinations (equivalently, maximum and minimum observed output).

**Numerical Observation 4.1.** The yield curve at the top of the cycle lies uniformly below the yield curve at the bottom of the cycle. The same relationship is generally observed also for states between these polar values.

**Numerical Observation 4.2.** The yield curve is rising at the top of the cycle and falling at the bottom of the cycle.

**Numerical Observation 4.3.** The average yield curve \( r_n \) is upward-sloping for all parameter choices.

These results have natural interpretations within the structure of this simple model. Turning first to Observation 4.1, we notice that consumption is particularly high at the top of the cycle, when output achieves its highest level. Moreover, as a consequence of the ergodic property of the consumption series, and with less than perfect shock correlation, it is to be expected that consumption will, on average, be lower in the future. Being consumption smoothers, agents will consequently save more, thereby driving down the equilibrium rates at the top of the cycle. At the bottom of the cycle consumption on average will rise as time passes, thus reducing agents' need to save; relatively higher equilibrium interest rates are the natural result of this reduced savings.

Related reasoning applies to Observation 4.2. The interest rates reflect, essentially, the average rate of growth in consumption over the respective future time periods. Looking to the future from the bottom of the cycle, the average growth rate will be positive but decreasing with the horizon. This initial high growth in consumption followed by progressively lower growth is perfectly reflected in the declining yield curve. Analogous reasoning explains the rising yield curve at the top of the cycle. What is especially interesting about Observation 4.2 is the monotonicity it implies.

Somewhat weaker versions of Observations 4.1 and 4.2 can be proven analytically as a pure consequence of the asymptotic behavior of the state variables in our economy. This is the substance of Propositions 4.1 and 4.2

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11 We do not claim, however, that this is a realistic model of the business cycle.
which follow:

**Proposition 4.1.** For sufficiently long maturities, the yield curve at the top of the cycle lies uniformly below the yield curve at the bottom of the cycle.

**Proof.** Let the range of the stationary capital stock–shock to technology distribution be denoted by \([k, \overline{k}] \times [\underline{\lambda}, \overline{\lambda}]\). Since, for this class of models, consumption is an increasing function of the state variables, the following inequality holds for any \(n > 0\):

\[
\frac{u'(c(k_{t+n}, \lambda_{t+n}))}{u'(c(\underline{k}_t, \underline{\lambda}_t))} < \frac{u'(c(k_{t+n}, \lambda_{t+n}))}{u'(c(\overline{k}_t, \overline{\lambda}_t))},
\]

for all

\((k_{t+n}, \lambda_{t+n}) \in [k, \overline{k}] \times [\underline{\lambda}, \overline{\lambda}]\).

Since the conditional distributions \(H(k_{t+n}, \lambda_{t+n} | k_t, \lambda_t)\) and \(H(k_{t+n}, \lambda_{t+n} | \overline{k}_t, \overline{\lambda}_t)\) have the same limit as \(n \to \infty\), for \(n\) sufficiently large:

\[
\beta^n \int \frac{u'(c(k_{t+n}, \lambda_{t+n}))}{u'(c(\underline{k}_t, \underline{\lambda}_t))} H(k_{t+n}, \lambda_{t+n} | k_t, \lambda_t)
< \beta^n \int \frac{u'(c(k_{t+n}, \lambda_{t+n}))}{u'(c(\overline{k}_t, \overline{\lambda}_t))} H(k_{t+n}, \lambda_{t+n} | \overline{k}_t, \overline{\lambda}_t),
\]

and thus

\(P_n(k_t, \lambda_t) < P_n(\overline{k}_t, \overline{\lambda}_t)\).

Hence,

\(r_n(k_t, \lambda_t) > r_n(\overline{k}_t, \overline{\lambda}_t)\).

**Proposition 4.2.** For sufficiently long maturities, yields on long bonds must exceed the one-period rate at the top of the cycle. The opposite is true at the bottom of the cycle.
Proof. This is seen as a manipulation of eq. (3):

\[ r_n - r_1 = \frac{1}{\beta} \left[ \left( \frac{u'(c(k_t, \lambda_t))}{\int u'(c(k_{t+n}, \lambda_{t+n}))H_n(dk_{t+n}, d\lambda_{t+n} | k_t, \lambda_t)} \right)^{1/n} - \frac{u'(c(k_t, \lambda_t))}{\int u'(c(k_{t+1}, \lambda_{t+1}))H_1(dk_{t+1}, d\lambda_{t+1} | k_t, \lambda_t)} \right]. \tag{8} \]

As \( n \to \infty, \)

\[ r_n - r_1 \overset{\text{a.s.}}{\to} \frac{1}{\beta} \left[ 1 - \frac{u'(c(k_t, \lambda_t))}{\int u'(c(k_{t+1}, \lambda_{t+1}))H_1(dk_{t+1}, d\lambda_{t+1} | k_t, \lambda_t)} \right]. \tag{9} \]

If we view the top of the cycle as a consumption level \( c(k_t, \lambda_t) \) for which \( u'(c(k_t, \lambda_t)) < \int u'(c(k_{t+1}, \lambda_{t+1}))H(dk_{t+1}, d\lambda_{t+1} | k_t, \lambda_t), \) then \( r_n - r_1 > 0 \) for \( n \) sufficiently large. Analogous comments apply to the bottom of the cycle; i.e., for sufficiently large \( n, \) \( r_n - r_1 < 0. \)

It is difficult to rationalize Observation 4.3 with simple intuition, especially in the light of 4.2 which states that the term structure is rising for some capital stock–shock pairs and falling for others. In this model (as we shall see), the term structure is, however, asymptotic to the same strictly positive limit for all states. Since short rates are substantially negative for many states, these facts together suggest a general upward bias. As will be further discussed in section 7, this does not necessarily imply, however, that future one-period rates will exceed today’s one-period rate. This is consistent with the findings of, e.g., Shiller et al. (1983) who conclude that implied forward rates provide poor forecasts of future spot rates.

In summary, Observations 4.1 and 4.2 appear substantially inconsistent with the observed behavior of the nominal term structure and with the behavior of the observed real term structure as well. This issue clearly deserves greater theoretical and empirical study. In particular, the role played by inventory smoothing in possibly reversing this countercyclical behavior needs to be addressed.
Our focus now turns to a brief quantitative assessment of the model's performance.

4.2. A quantitative assessment

For the U.S. economy over the period 1926–1977, Ibbotson and Sinquefeld (1979) report that the geometric mean real return on T-bills was 0.0% with a standard deviation of 4.6%. These authors also report, for the same historical period, that holding returns on long-term government bonds averaged 0.7% with a standard deviation of 6%.

To compare these statistics with the output of our simple dynamic model, we must choose parameter values which in some reasonable sense approximate their real world counterparts. Preferably, their determination should be the result of independent micro studies (cross-model verification). An assemblage of such information has already been performed for the real business cycle literature and excellent summaries may be found in Prescott (1986), Kydland and Prescott (1982), and Hansen (1985). Following these authors, we choose $\beta = 0.96$, $\gamma = 0.33$, and $\alpha = 0.36$. The choice of $\alpha$ value, for example, gives income shares to capital and labor of, respectively, 36% and 64%, proportions which closely reflect what is observed for the U.S. economy. The parameter $\beta$ is set at 0.96 to give the observed 4% average real return to tangible capital. The shock structure parameters $\pi$ and $\Delta$ are somewhat more problematic, as the random processes employed in the aforementioned studies differ substantially from the one presented here. It is possible via a simple calculation, however, to determine values for $\pi$ and $\Delta$ for which the mean and variance of our shock process are identical to those of, e.g., Kydland and Prescott (1982). This is what we do, obtaining estimates for $\Delta$ and $\pi$ of, respectively, 0.03 and 0.54. Using these parameter values, the term structure presented in table 1 results.

On average, the model's prediction of average short-term (one-period) rates is too large and its estimate of variation too small. Similar observations apply to maturities further out in the term structure. Indeed, if we hypothesize that the data upon which Ibbotson and Sinquefeld (1979) statistics are based was drawn from a stationary distribution, it would seem especially
consistent for long-term real rates to average less than 1% while the ‘forecast’ of them – the term structure – consistently averaged more than 4%, roughly six times greater. Thus, we are forced to infer that the actual real term structure, on average, must, for all maturities, lie below that generated by the model. At least for the U.S., the model appears to quantitatively err by a large margin.14,15

In view of earlier studies, this fact is not entirely surprising. Indeed, for the same class of models, Mehra and Prescott (1985) find that the aggregate risk premium is too small unless the representative agent’s coefficient of relative risk aversion is set at some unrealistically high level. A substantial reduction in the reported return levels can be achieved, however, by increasing the shock variation. To demonstrate this fact, we undertook to calculate the average real term structure for $\Delta = 0.5$ in table 2, with all other parameters set at levels identical to those reported in table 1.

14 Less so for the Israeli economy. Cukierman (1981) reports short-term real rates in the range of 0.2% to 6.2%, with a mean value of 4.4% and a standard deviation of 3.5%. We have no information, however, that our choice of parameters is appropriate to that society.

15 As noted earlier, the 4% figure is a good approximation to the observed historical return to tangible capital, however. See Prescott (1986).
Even when we postulate shocks of this totally unrealistic magnitude, we observe that the improvement in average real rates is confined to short maturities alone, at the expense of excess variation. Thus we are forced again to conclude that this simple model is incapable of explaining the real term structure quantitatively.

For long rates, the excessive levels generated by the model are due to the dominance of the subjective discount parameter $\beta$ in the determination of these values. This is evident from the following proposition.

**Proposition 4.3.** Let $[k, \bar{k}] \times [\lambda, \bar{\lambda}]$ denote the range of the joint stationary distribution on capital and the shock to technology. (i) For any $(k, \lambda) \in [k, \bar{k}] \times [\lambda, \bar{\lambda}]$, the conditional term structure is bounded above and below. (ii) Furthermore, for every $(k, \lambda)$, the asymptotic limit is $1/\beta - 1$. The average term structure thus has the same limit as well.

**Proof.** (i) For any $(k, \lambda) \in R^k \times [\lambda, \bar{\lambda}]$, the following inequalities are satisfied:

$$
\beta^n \frac{u'(c(k, \lambda))}{u'(c(k, \lambda))} \leq \beta^n \int \frac{u'(c(k_{t+n}, \lambda_{t+n}))}{u'(c(k, \lambda))} H(dk_{t+n}, d\lambda_{t+n} | k, \lambda)
$$

$$
\leq \beta^n \frac{u'(c(k, \lambda))}{u'(c(k, \lambda))}.
$$

Therefore,

$$
\frac{1}{\beta} \left( \frac{1}{u'(c(k, \lambda))} \right)^{1/n} - 1 \leq r_n(k, \lambda) \leq \frac{1}{\beta} \left( \frac{1}{u'(c(k, \lambda))} \right)^{1/n} - 1,
$$

and hence

$$
\frac{1}{\beta} \left( \frac{u'(c(k, \lambda))}{u'(c(k, \lambda))} \right)^{1/n} - 1 \leq r_n(k, \lambda) \leq \frac{1}{\beta} \left( \frac{u'(c(k, \lambda))}{u'(c(k, \lambda))} \right)^{1/n} - 1.
$$

(ii) From the equation above we see that both upper and lower bounds converge to $1/\beta - 1$ as $n \to \infty$.\(^{16}\)

\(^{16}\)Theorem 4.1 allows us to address one 'loose end' regarding our quantitative assessment. This result clearly suggests that rates on long bonds will be dominated by $1/\beta - 1$. Thus, could not the model's quantitative output be improved if we were to choose $\beta = 0.01$? This would be so except that we would need to interpret the model's period as representing three months, if we were to be consistent with microstudies [see Prescott (1986)]. On an annualized basis the same overestimate would naturally remain.
Proposition 4.3 is a powerful result. In particular it suggests that the properties of long rates will be more or less unaffected by any of the model's parameters except $\beta$ (for bonds of sufficiently long term). Thus it is not surprising that the enormous increase in the shock variance underlying table 2 did little to alter average long rates. Proposition 4.3 underlies many more model properties. It is to these properties that we now turn.

5. Volatility issues

This model is, by construction, informationally efficient [see Lucas (1978)]. In this section we examine the implications of this efficiency for the relative volatility of short- and long-term rates and one-period holding returns and compare these results with what has appeared in the literature. To facilitate this discussion, and those that follow, we first propose the following additional notation:

\[
\begin{align*}
HP_{t,n}^k &= \text{`annualized' holding period return over } k \text{ periods on an } n\text{-period maturity bond,} \\
r_{t,n} &= \text{yield to maturity on an } n\text{-period bond purchased at time } t, \\
f_{t,k}^n &= n\text{-period forward rate, } k \text{ periods forward,} \\
FP_{t,k}^n &= f_{t,k}^n - E_t(r_{t+k,n}), \text{ the forward premium.}
\end{align*}
\]

With respect to volatility, the literature is once again unambiguous: nominal long rates vary less than nominal short rates. The same is true for our model of real rates. Again, this may be seen as a consequence of the ergodic property of the consumption process and is consistent with the predictions of other rational expectations models of nominal rates [see, e.g., Shiller (1979, pp. 1190–1194) and Salyer (1989)].

**Proposition 5.1.** There exists an $N$ such that for all $n \geq N$, \(\text{Std.dev. } r_n(k, \lambda) < \text{Std.dev. } r_i(k, \lambda)\).

**Proof.** This follows from three facts:

(i) \(\text{Std.dev. } r_i(k, \lambda) > 0\).

(ii) \(r_n(\overline{k}, \overline{\lambda}) \leq r_n(k, \lambda) \leq r_n(\underline{k}, \underline{\lambda})\), for all \((k, \lambda)\) in the stationary range. Here \((\overline{k}, \overline{\lambda})\) and \((\underline{k}, \underline{\lambda})\) denote, as before, the top and bottom of the cycle, respectively.

(iii) \(\lim_{n \to \infty} r_n(\overline{k}, \overline{\lambda}) = \lim_{n \to \infty} r_n(\overline{k}, \overline{\lambda}) = 1/\beta - 1\) (Proposition 4.3). 

The results of our numerical investigation provide an attractive strengthening of this result.
Table 3

\[\alpha = 0.36, \beta = 0.96, \gamma = 0.33, \tau = 0.24, \Delta = 0.03.\]

<table>
<thead>
<tr>
<th></th>
<th>Std.dev. (HP_{t,n}^1)</th>
<th>Std.dev. (r_{t,n})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.24</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.0331</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.0384</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.0401</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.0407</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.0409</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>0.0409</td>
<td></td>
</tr>
</tbody>
</table>

**Numerical Observation 5.1.** For all choices of parameter values, the standard deviation of the stationary distribution of the yield to maturity is seen to decline with maturity.

It is frequently claimed that market efficiency requires that the volatility of one-period nominal holding period returns on long bonds should be less than the volatility of short rates. Yet, this is not observed empirically [see, again, Shiller (1979)]. Neither is it observed in this model of real rates. To illustrate, table 3 compares the standard deviation of the one-period holding return \(HP_{t,n}^1\) on an \(n\)-period bond with that of the one-period interest rate \(r_{t,1}\) for a representative parameter set.

Our model thus performs, qualitatively, quite well along the volatility dimension.

**6. Correlation tests of market efficiency**

From the perspective of standard efficient market theory, price changes and holding period returns should be uncorrelated with any variables in agent’s information sets. This set would include past holding period returns and price changes as well as any other statistical variables. The extent to which this property carries over into our model is the subject of the current section.

To begin, it has been asserted that in an efficient market successive holding period returns should be uncorrelated with past holding period returns and rate spreads. Campbell and Shiller (1984), in particular, correlated today’s excess one-period holding return on long bonds above the short interest rate with the spread between the long and the short interest rate \(- \text{corr}[HP_{t,n}^1 - r_{t,1}, r_{t,n} - r_{t,1}]\). That is, what does the slope of the yield curve have to say about the *ex post* premium to holding long bonds? By performing the

---

17 More precisely, these authors perform the equivalent correlations between the forward expected spot rate differential and the long–short spread.
Table 4
\[ \text{corr}(HP_{t,n}^{1} - r_{t,1}, r_{t,n} - r_{t,1}). \]
\[ \alpha = 0.25, \beta = 0.95, \gamma = -1, \Delta = 0.5. \]
\[
\begin{array}{ccc}
\pi = 0.33 & \pi = 0.9 \\
\hline
n = 2 & -0.806 & 0.329 \\
n = 3 & -0.949 & -0.0338 \\
n = 4 & -0.968 & 0.0242 \\
n = 5 & -0.974 & 0.0804 \\
n = 10 & -0.983 & 0.0668 \\
n = 15 & -0.905 & -0.302 \\
\end{array}
\]

Table 5
\[ \text{corr}(r_{t+j,n} - r_{t+j-1,n}, r_{t,n} - r_{t-1,n}). \]
\[ \alpha = 0.25, \beta = 0.95, \gamma = 0.7, \Delta = 0.5. \]
\[
\begin{array}{cccc}
\gamma = 0.5 & \gamma = -1 \\
\hline
j = 1 & n = 1 & n = 15 & n = 1 & n = 15 \\
j = 2 & 0.187 & 0.794 & 0.213 & 0.823 \\
& 0.032 & 0.494 & 0.163 & 0.560 \\
\end{array}
\]

analogous computation in our idealized setting, we find this correlation generally to be negative. It turns positive, however, when shock persistence is very high. This is illustrated in table 4, again for a representative parameter set.

These results are consistent with the general expectations hypothesis that the interest rate is more likely to rise the steeper the yield curve. For our economy, upward-sloping and downward-sloping yield curves, respectively, correspond to a historically low or high interest rate level [recall eqs. (8) and (9)]. These correlations, therefore, are also consistent with the traditional Keynesian view. Campbell and Shiller (1984) had difficulties, however, in deriving such a negative correlation and suggested the possibility of a ‘perverse’ relationship.

More traditional studies [e.g., Fama (1975)] have noted that, while rate levels are highly correlated, interest rate changes are uncorrelated. In general this phenomenon is not confirmed in the model. Table 5 compares the correlations of successive and two-period lagged rate changes for bonds of one- and fifteen-year maturities. What is especially striking is the high degree of persistence in rate changes on long bonds. Once again, this is purely a result of the ergodic property of the model and is independent of the precise parameter values: Since so much of the yield to maturity on long bonds is due to events in
the distant future – the probability of which is unaffected by events today (ergodicity) – then yields change little over the business cycle, thus giving rise to high correlations. That short interest rates are more closely related to the cycle than long rates – or that long rates ‘underreact’ to changes in the short rate – is indicated by numerous studies [e.g., Campbell and Shiller (1984) and Mankiw and Summers (1984)]. Here, unlike what seems to be the claim in some of these studies, this property cannot be taken as an indication of a failure of a rational expectations theory of the interest rate [see also the comments by Weiss (1984)].

The power of the risk aversion parameter to smooth consumption is also evidenced by the fact that, for the $\gamma = -1$ case, the two-period lagged rate changes are still very high. We wish to point out, however, that these earlier studies which reported negligible correlation dealt with measured changes over intervals of time which are relatively much shorter than those of this model (judging from the magnitude of the productivity shocks and the size of the discount factor $\beta$).

The high degree of persistence in the long rates should also produce persistence in their excess holding returns. This is confirmed by table 6 where we have computed the serial correlation $\text{corr}(HP_{t+j,n}^1 - r_{t+j,1}, HP_{t,n}^1 - r_{t,1})$ for different maturities and parameter values. We note the very high correlation when the shock persistence is high.

Thus, on average for this model, investors excess returns today on the ownership of long bonds will, to a significant degree, be preserved in the future. Once again this is purely the result of the model’s ergodic feature and is independent of the particular choice of parameter values. In fact, as the maturity period tends to infinity (as $n$ gets large in table 6), the correlation will approach one, irrespective of the parameter values. (This was pointed out to us by a referee.)
Table 7
The structure of the forward premium.
Panel A: $\alpha = 0.25, \beta = 0.95, \gamma = -1, \pi = 0.5, \Delta = 0.5$;
pool B: $\alpha = 0.25, \beta = 0.95, \gamma = -2, \pi = 0.9, \Delta = 0.5$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$E(f_{1,n}^1)$</th>
<th>$E(r_{1+n,1})$</th>
<th>$E(FP_{1,n}^1)$</th>
<th>$F(f_{1,1}^n)$</th>
<th>$F(r_{1+n,1})$</th>
<th>$F(FP_{1,n}^n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-0.9%$</td>
<td>13.0%</td>
<td></td>
<td>$-0.9%$</td>
<td>$-13.9%$</td>
<td>13.0%</td>
</tr>
<tr>
<td>5</td>
<td>5.2%</td>
<td>$-13.9%$</td>
<td>19.1%</td>
<td>3.2%</td>
<td>1.7%</td>
<td>4.9%</td>
</tr>
<tr>
<td>10</td>
<td>5.3%</td>
<td>19.2%</td>
<td></td>
<td>4.1%</td>
<td>1.3%</td>
<td>2.8%</td>
</tr>
<tr>
<td>15</td>
<td>5.3%</td>
<td>19.2%</td>
<td></td>
<td>4.4%</td>
<td>2.5%</td>
<td>1.9%</td>
</tr>
</tbody>
</table>

Panel B

<table>
<thead>
<tr>
<th>$n$</th>
<th>$E(f_{1,n}^1)$</th>
<th>$E(r_{1+n,1})$</th>
<th>$E(FP_{1,n}^1)$</th>
<th>$F(f_{1,1}^n)$</th>
<th>$F(r_{1+n,1})$</th>
<th>$F(FP_{1,n}^n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>2.3%</td>
<td></td>
<td>$-17.66%$</td>
<td>$-19.9%$</td>
<td>2.3%</td>
</tr>
<tr>
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<td>0.5%</td>
<td>$-19.9%$</td>
<td>19.4%</td>
<td>$-8.3$</td>
<td>$-12.4%$</td>
<td>3.9%</td>
</tr>
<tr>
<td>10</td>
<td>4.2%</td>
<td>24.1%</td>
<td></td>
<td>$-3.1%$</td>
<td>$-5.8%$</td>
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</tr>
<tr>
<td>15</td>
<td>5.0%</td>
<td>25.0%</td>
<td></td>
<td>$-0.6%$</td>
<td>$-2.6%$</td>
<td>2.0%</td>
</tr>
</tbody>
</table>

In summary, the absence of correlation in holding period returns and rate changes that is frequently cited as evidence of market efficiency is simply not observed in our model. To ensure such independence would apparently require a more specialized market structure than the one presented here. Indeed, the message of these results is that investors must use the information present in such correlations if they are to earn returns consistent with the level of risk they are bearing.

7. Issues of information

All the literature to date [see Backus et al. (1989) for an excellent summary] points to the conclusion that nominal forward rates are biased forecasts of future spot rates and that the degree of the bias (the forward premium) varies through time. Our model’s qualitative results are largely consistent with these findings.

As the average yield curve is rising for this model (Numerical Observation 4.3), forward rates, on average, must provide biased forecasts of future spot rates (this is because the expected spot rate is constant in a stationary economy). Equivalently, the average forward premium will be positive. The more complete picture is as follows: (1) the longer the forecast period (and the larger the volatility of the forecasted interest rate), the larger the forward premium, and (2) the longer the maturity of the interest rate forecasted (and the smaller its volatility), the smaller the forward premium. These results apply for all parameter values. It is also interesting to note the effect, presented in panel B
Table 8

The stability of the forward premium.

Panel A: $\alpha = 0.25$, $\beta = 0.95$, $\gamma = -1$, $\pi = 0.5$, $\Delta = 0.5$.
Panel B: $\alpha = 0.25$, $\beta = 0.95$, $\gamma = -2$, $\pi = 0.9$, $\Delta = 0.5$.

<table>
<thead>
<tr>
<th>Bottom</th>
<th>Middle</th>
<th>Top</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{t,n}$</td>
<td>$E_j(r_{t+n,1})$</td>
<td>$F_{P_{t,n}}^j$</td>
</tr>
<tr>
<td>$f_{t,n}$</td>
<td>$E_j(r_{t+n,1})$</td>
<td>$F_{P_{t,n}}^j$</td>
</tr>
<tr>
<td>$f_{t,n}$</td>
<td>$E_j(r_{t+n,1})$</td>
<td>$F_{P_{t,n}}^j$</td>
</tr>
</tbody>
</table>

Panel A

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 1$</td>
<td>52.6%</td>
<td>35.4%</td>
</tr>
<tr>
<td>$n = 5$</td>
<td>6.7%</td>
<td>-12.8%</td>
</tr>
<tr>
<td>$n = 10$</td>
<td>5.3%</td>
<td>-13.9%</td>
</tr>
<tr>
<td>$n = 15$</td>
<td>5.3%</td>
<td>-13.9%</td>
</tr>
</tbody>
</table>

Panel B

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 1$</td>
<td>15.9%</td>
<td>15.1%</td>
</tr>
<tr>
<td>$n = 5$</td>
<td>13.6%</td>
<td>0.0%</td>
</tr>
<tr>
<td>$n = 10$</td>
<td>10.2%</td>
<td>-11.1%</td>
</tr>
<tr>
<td>$n = 15$</td>
<td>7.8%</td>
<td>-16.0%</td>
</tr>
</tbody>
</table>

of table 7, of a simultaneous increase in the economy’s shock persistence and risk aversion. While the former change should lead to a decrease in the volatility of future interest rates, the latter change should increase the impact of a given volatility on the forward premium. We see that the former effect dominates for the shortest forecast span and maturity, while the latter dominates for the longest forecast spans. The net effect is thus to widen the spread of the forward premia across the maturity range.

The usefulness of the yield curve information regarding the future expected interest rates obviously depends upon the stability across time of the forward premia: if these premia are reasonably stable, then an adjustment of forward rates for their average values should yield useful forecasts of future interest rates. Table 8 provides us with an indication of this stability within our model economy. Here we provide (conditional) yield curve information at, respectively, the ‘top’, ‘bottom’, and ‘middle’ of the cycle (the middle of the cycle represents the median capital stock—shock combination). This table gives forward and expected one-period interest rates and the forward premium $F_{P_{t,n}}^j = f_{t,n} - E_j(r_{t+n,1})$ for various time periods ahead. We record the following observations, some of which are immediate extensions of previous results:

**Numerical Observation 7.1.** In general, the forward premium is positive for all time periods and all points on the cycle.

**Numerical Observation 7.2.** With low shock persistence and low risk aversion, the forward premium for time periods distant in the future is stable over the cycle.
Table 9

\[
\text{corr}(r_{t+n+1,1} - r_{t+n,1}, f_{t,n+1}^1 - f_{t,n}^1).
\]
\[
\alpha = 0.25, \beta = 0.95, \gamma = -1, \pi = 0.5, \Delta = 0.5.
\]

<table>
<thead>
<tr>
<th>(n)</th>
<th>(0.763)</th>
<th>(0.129)</th>
<th>(0.089)</th>
<th>(0.030)</th>
<th>(0.013)</th>
<th>(0.000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>2</td>
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<tr>
<td>3</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For the immediate future, however, the premium varies considerably and positively with the level of interest rates. The premium is high when interest rates are high (at the ‘bottom of the cycle’) and low when rates are low (at the ‘top of the cycle’).

Numerical Observation 7.3. Higher shock correlation and greater risk aversion produce less stability in the forward premium at the short end of the term structure, and also reverse the relationship between the forward premium and the level of interest rates. Thus, the premium for all periods is low when rates are high and high when rates are low.

Thus, depending on the parameter choice, our model seems to accommodate both traditional, competing theories of the comovement of interest rates and forward premia. According to the Kessel (1965) liquidity interpretation of the forward premium, there should be a positive relationship between interest rates and forward premia as displayed by panel A of table 8. The Keynes-Hicks insurance view of the forward premium, on the other hand, predicts an inverse relationship with the interest rate level as displayed by panel B of table 8.

We should also note that, while the panel A contradicts the result of Startz (1982), this is not so for panel B. Based on a simple version of a rational expectations structure and using one-month T-bill data, he derives estimates of forward premium volatilities which increase monotonically with the length of the forecast span. The issue of forward premium obviously deserves further theoretical attention; in particular, the role played by risk aversion.

Fama (1984) suggests interest rate forecasts based upon forward rate differences as a possible way around the problem of highly volatile forward premia, at least for short-term forecasting. He derives a surprisingly high correlation between the one-month forward-spot T-bill rate and the following month’s change in the one-month rate. The analogous correlations \[
\text{corr}(r_{t+n+1,1} - r_{t+n,1}, f_{t,n+1}^1 - f_{t,n}^1)
\]
between later changes in this rate and the current spread between corresponding forward rates fall off dramatically,
The stability of the forward premium.

\[ \alpha = 0.36, \beta = 0.96, \gamma = 0.33, \tau = 0.54, \Delta = 0.03. \]

![Table 10](attachment:image.png)

However, as the forecast horizon increases. It is interesting to note that the identical phenomena is observed in our model. Table 9 presents these results for a representative parameter set.

Table 8 obviously illustrates very substantial variability in the one-period forward premium over the cycle. This is due in large measure to the choice of parameters, and, in particular, the large shocks. If we again choose parameter values which can reasonably be argued to hold for the U.S. economy, however, the picture changes. This is the substance of table 10.

Note that the forward premium is quite stable over the cycle, and is, in fact, too stable to cause a rejection of the expectations hypothesis for long bonds. This is consistent with the findings of Backus et al. (1989). While qualitatively very flexible, once again the model would appear to fail quantitatively.

### 8. Concluding comments

In this paper we have analyzed a simple real business cycle model with regard to its ability to replicate the qualitative and quantitative properties of the term structure of real interest rates. Our efforts have been hampered by the relative dearth of empirical studies of real interest rate behavior for bonds of long maturity (up to fifteen years) – a fact that has forced us to undertake our comparisons with studies of nominal rate behavior. Given this caveat, we summarize the model's performance in two steps:

1. In a qualitative sense, the model does very well along most dimensions (volatility of short versus long rates, for example) and very poorly along a few others. In particular, the high correlation in model rate changes and holding period returns is not present in real world data. This is due to the strong ergodic feature of the model. Least satisfying, perhaps, are our results concerning the behavior of the yield curve over the cycle. A worthwhile focus of further research would be to improve the model's performance in this regard.
(2) Quantitatively, the model is not successful. In particular, long rates are too dominated by the time preference parameter (Proposition 4.3) to give reasonable level estimates.

Our conclusions are consistent with those of other authors, in particular Mehra and Prescott (1985) and Backus et al. (1989).

Appendix

To illustrate this procedure, define the set $T_{ij}$ by

$$T_{ij} = \{(k_i, \lambda_j) : k_i \in \Gamma, \lambda_j \in \{\frac{1}{2}, 1, \frac{3}{2}\}\}.$$  \hfill (A.1)

For each $(k_i, \lambda_j) \in T_{ij}$ first compute the table $T_0$ of values where

$$T_0 = \{V_0(k_i, \lambda_j) : V_0(k_i, \lambda_j) = u(f(k_i) \lambda_j), (k_i, \lambda_j) \in T_{ij}\}.$$  \hfill (A.2)

In the next stage of this process, construct two tables, the first of which, $T_1$, being defined by

$$T_1 = \left\{ V_i(k_i, \lambda_j) : V_i(k_i, \lambda_j) = \max_{s(f(k_i) \lambda_j) \in \Gamma} \left\{ u(f(k_i) \lambda_j - s(k_i, \lambda_j)) \right\} \right\} \times (k_i, \lambda_j) \in T_{ij} \right\}.$$  \hfill (A.3)

The second table, $S_1$, records the optimal savings level which solves $V_i(\cdot, \cdot)$; i.e.,

$$S_1 = \left\{ s_i(k_i, \lambda_j) : V(k_i, \lambda_j) \in T_i, s_i(k_i, \lambda_j) \right\}.$$  \hfill (A.4)

By repeating this process over and over again we were able to define the
optimal savings policy $s^*(k_i, \lambda_j)$ for all $(k_i, \lambda_j) \in T_{ij}$ as

$$s^*(k_i, \lambda_j) = \lim_{n \to \infty} s_n(k_i, \lambda_j);$$

(A.5)

analogously, the optimal consumption function is given by $c^*(k_i, \lambda_j) = f(k_i, \lambda_j) - s^*(k_i, \lambda_j)$.

Although this procedure defines the optimal policy functions over the relevant range, it does not identify the set of possible stationary capital stock values. This latter task was accomplished by actually constructing the time path of the economy for 50,000 periods and observing the capital stock values assumed. First a sequence of 50,000 shock values $\{\lambda_j\}_{j=0}^{49,999}$ was generated in such a way as to reflect the selected choice of transition matrix. After arbitrarily choosing the initial capital stock level of $k_0 = \frac{1}{2}$, the corresponding capital stock sequence was generated according to $k_0 = \frac{1}{2}$, $k_{t+1} = s^*(k_t, \lambda)$. To be certain the joint process had entered its stationary state, the first 10,000 entries in the two sequences were dropped; thus $S(K, \lambda) = \{(k_i, \lambda_j)\}_{i=10,001}^{49,999}$ was retained. Using this sequence $S(K, \lambda)$, we defined the set $K = \{k_1, k_2, \ldots, k_n\}$, $k_i \in \Gamma$, as the set of capital stock levels appearing in $S(K, \lambda)$ ranked from lowest to highest. This set $K$ thus becomes our approximation to the stationary range on capital, and sufficient information is now available to construct the one-step state transition matrix $\hat{\pi}$ with entries $\hat{\pi}(i,j),(s,w)$:

$$\hat{\pi} = \begin{bmatrix}
(\hat{k}_1, \lambda_1) & \cdots & \cdots & \cdots \\
(\hat{k}_1, \lambda_2) & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
(\hat{k}_i, \lambda_j) & \cdots & \cdots & \hat{\pi}_{(i,j),(s,w)} \\
(\hat{k}_n, \lambda_3) & \cdots & \cdots & \cdots \\
\end{bmatrix}.$$

(A.6)

The entries $\hat{\pi}(i,j),(s,w)$ which give $\text{Prob}(k_{t+1}, \lambda_{t+1}) = (\hat{k}_s, \lambda_w) | (k_t, \lambda_j) = (\hat{k}_i, \lambda_j)$ are governed by the optimal savings policy in the following way:

$$\hat{\pi}_{(i,j),(s,w)} = \begin{cases} 
0 & \text{if } s^*(\hat{k}_i, \lambda_j) \neq \hat{k}_s, \\
\pi_{jw} & \text{if } s^*(k_i, \lambda_j) = \hat{k}_s.
\end{cases}$$

(A.7)
The general formula for pricing a pure discount bond must first be obtained. In this setting a pure discount bond issued in state \((\hat{k}_t, \lambda_j)\) and paying one unit of consumption in period \(n\) irrespective of the state must, in equilibrium, be priced according to

\[
P(\hat{k}_i, \lambda_j) = \beta^n \sum_{(k_s, \lambda_w) = (\hat{k}_i, \lambda_j)} \left[ \frac{u'(c^n(\hat{k}_s, \lambda_w))}{u'(c^n(k_t, \lambda_j))} \right] \pi_{(i,j),(s,w)}^n,
\]

(A.8)

These are the 'conditional' bond prices. For every state \((\hat{k}_i, \lambda_j) \in \hat{K} \times \{\frac{1}{2}, 1, \frac{3}{2}\}\), the analogous state dependent term structure (we computed it for 15 time periods) is thus defined by the sequence

\[
\left\{ \frac{1}{P_1(\hat{k}_i, \lambda_j)} - 1, \frac{1}{P_2(\hat{k}_i, \lambda_j)} - 1, \frac{1}{P_3(\hat{k}_i, \lambda_j)} - 1, \ldots, \left( \frac{1}{P_{15}(\hat{k}_i, \lambda_j)} \right)^{1/15} - 1 \right\}.
\]

To compute average (across all states) bond prices, it is first necessary to determine the limiting relative frequency of the various \((\hat{k}_i, \lambda_j)\) pairs. The relative frequency with which the individual \((\hat{k}_i, \lambda_j)\) pairs appears in the sequence \(S(K, \lambda)\) reasonably approximates this true distribution. Thus, the average price of an \(n\)-period pure discount bond was approximated according to

\[
P_n = \frac{1}{40,000} \sum_{(\hat{k}_i, \lambda_j) \in S(K, \lambda)} P_n(\hat{k}_i, \lambda_j).
\]

(A.9)

The term structure as computed from average bond prices \(\{\hat{P}_n\}\) was then determined according to \(\{1/\hat{P}_1 - 1, (1/\hat{P}_2)^{1/2} - 1, (1/\hat{P}_3)^{1/3} - 1, \ldots, (1/\hat{P}_{15})^{1/15} - 1\}\). See footnote 7. Alternatively, the term structure as the average of conditional interest rates \(\{r_n\}\) was derived as

\[
r_n = \frac{1}{40,000} \sum_{(\hat{k}_i, \lambda_j) \in S(K, \lambda)} \left[ \frac{1}{P_{n_r}(\hat{k}_i, \lambda_j)} \right]^{1/n} - 1.
\]

It is these latter values that are reported in the text.
References


Hicks, J., 1946, Value and capital, 2nd ed. (Oxford University Press, London).


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