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Comparative Dynamics of an Equilibrium Intertemporal Asset Pricing Model

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This paper uses recursive competitive theory to develop a general equilibrium asset pricing model. In this framework all prices and rates of return are endogenously determined, thus enabling us to analyze the effects of changes in preferences, technological uncertainty, and expectations on the structure of security prices. In particular we focus on how the market risk premium varies with changes in the underlying economic environment, an issue which other asset pricing models have chosen not to address.

1. INTRODUCTION

Capital market theory has made significant advances over the last fifteen years. In particular, the pioneering two parameter capital asset pricing model of Sharpe (1964), Lintner (1965), and Mossin (1966) and its subsequent generalizations and extensions provide the first rigorous theory of the allocation and pricing of risk.[1] The appeal of this literature is further enhanced by the testable hypothesis it conveniently provides.

A major shortcoming of these models, however, is that they assume an exogenously specified returns generating process; that is, unlike in the Arrow–Debreu tradition, the security prices and returns are not shown to arise from the interactions of profit maximizing firms and utility maximizing individuals. With the link between equilibrium security prices and the underlying technology and preferences thus effectively broken, these models preclude any analysis of important issues such as the effects of alternative tax policies, technological shifts, changes in preferences or expectations on the structure of security prices.

This paper is an attempt to bridge the above-mentioned gap. It develops the micro-foundations of intertemporal asset pricing in a dynamic general equilibrium setting. Our work is in the tradition developed by Lucas and Prescott (1971). This tradition constructs simple, tractable, stationary, homogeneous agent economies, characterizes the efficient allocation, and supports this efficient allocation using a limited set of markets. In order to price securities with empirical counterparts, securities are viewed as vectors of elementary Arrow–Debreu securities and are so priced. In the spirit of this tradition, we provide a framework in which equilibrium prices, firm value, the market and risk free returns, and the market risk premium are explicitly linked to the underlying production technology, corporate decisions, and consumer preferences in a non-trivial way.

There is another rich theoretical tradition, most closely identified with the works of Radner (1972) and Hart (1975), which studies equilibrium in financial markets.

See Kreps (1979) for a lucid exposition. This approach is more general in that it allows for heterogeneous agents. Since the object of this paper is to study the comparative dynamics of asset pricing we restrict our analysis to the homogenous consumer recursive economy developed below.[2]

We consider an economy of rational agents—agents who behave optimally in light of their objectives. Firms maximize profits and consumers determine optimal consumption/investment policies which maximize their expected discounted utility subject to a budget constraint. All prices and price distributions are endogenous and are determined through market clearing, thereby ensuring consistency between corporate and individual decisions. Expectations are formed rationally; i.e. the prices and price distributions on which the economic agents base their consumption-investment-production decisions coincide with those implied by individual and corporate behaviour. In the sense of Fama (1970), Lucas and Prescott (1971), and Roll (1970), this means that the economy is informationally efficient—prices take into account (reflect) available information.[3] The paper exploits the Recursive Competitive Equilibrium Theory developed in Prescott and Mehra (1980) to ensure that the equilibrium allocation is Pareto optimal and that it may be characterized by a set of functions which simultaneously determine asset prices and optimal corporate behaviour. By exploiting the convergence results of Donaldson and Mehra (1983) we are further able to show that the behaviour of the economy over time may be characterized by its convergence to an appropriately defined stationary distribution.

A novel feature of the model is that we allow production shocks to be correlated through time, thereby enabling us explicitly to introduce the role of expectations. This is due to the fact that the current realization of the random shock provides information as to the likely future realizations. Expectations, we feel, significantly influence consumption and investment decisions. Since, in our model, equilibrium prices are linked to these decisions, our formulation provides a framework to examine the influence of expectations on risk premia and the risk structure of security prices.

To illustrate the framework of the paper, we also examine the effects of changes in risk aversion, expectations of economic uncertainty, and time preferences on the equilibrium market risk premium. (The market risk premium is defined as the expected return on the market less the risk free rate.)

It is clear that the Sharpe (1964)–Lintner (1965)–Mossin (1966) asset pricing model cannot address these issues since both the risk free rate and the expected return on the market are exogenous in that model. The works of Merton (1973) and Breeden (1979), while recognizing the fact that the return on the market and the risk free rate are functions of underlying state variables, are not concerned with how these returns vary as the underlying state variables change. Indeed, we are unaware of any model which examines changes in risk premia as a function of changes in the state variables and/or parameters of the economy.

When analysing expected returns, contemporary asset pricing models often neglect to differentiate explicitly between expected returns as conditioned by the current state, or (state independent) expected returns computed with respect to the joint stationary distribution of the underlying state variables. Yet this distinction has significant implications for empiricists concerned with the testing of these models. We clarify these distinctions and show what additional assumptions are required for the economy to converge to a stationary state (when production shocks are correlated). This latter analysis is necessary in order that unconditional expectations can be meaningfully defined (see Section 3).

Our framework is general enough to address a variety of issues in addition to those we have mentioned. In particular the effects of alternative tax policies, depreciation schemes, and changes in the ratio of output going to capital and labour on security prices and rates of return may be analysed. Since we consider a homogenous consumer economy and since the shock to technology is observed before production occurs, the unanimity issues addressed by Radner (1974), Grossman and Stiglitz (1976), Hart (1979) and Milne (1979) do not arise.

In addition to Brock (1979) and Cox, Ingersoll and Ross (1979) (see footnote 2), a number of other papers have appeared which address issues related to those considered here. Included among these are the works of Bhattacharya (1981), Breeden (1979), Constantinides (1981), Kanodia (1980), and Le Roy (1979ii). In all cases, however, these authors' objectives and modes of analysis differ substantially from our own.

The paper is divided into five sections. Section 2 provides a description of the economy and the appropriate equilibrium concept, while in Section 3 we formally derive explicit expressions for the prices of capital assets, and the risk free and market rates of return for our model context. Section 4 presents a numerical analysis of the effects of changes in consumer risk aversion or time preferences, and changes in the productivity shock structure on these returns and implied risk premium. Lastly, we provide a concluding commentary and relate our work to other aspects of the literature in Section 5.

2. THE ECONOMY

We consider a simplified (one good) version of the economy considered in Prescott and Mehra (1980).[4] The analytic foundation of our work will be Prescott and Mehra's (1980) observation that the investment and consumption policy functions arising as a solution to the (central planning) stochastic growth problem

$$\begin{aligned}(\mathrm{P}):\quad & \max E\left(\textstyle\sum_{t=0}^{\infty} \beta^t u(c_t)\right)\\ & \text{s.t. } c_t + k_{t+1} \leqq f(k_t)\lambda_t, \quad k_0 \text{ given}\end{aligned}$$

may be regarded as the aggregate investment and consumption functions arising from a decentralized homogeneous consumer economy in recursive competitive equilibrium. Problem (P) has the usual interpretation: preferences are time separable with period discount factor β and period utility function $u(\ )$ defined over period consumption c_t; k_t denotes capital available for production in period t while $f(\ )$ represents the period technology which is shocked by the stochastic factor λ_t. The shock sequence $\{\lambda_t\}$, $\underline{\lambda} \leqq \lambda_t \leqq \bar{\lambda}$, follows a Markov process with period transition density function $F(d\lambda_{t+1}, \lambda_t)$ and stationary cumulative distribution function $H(\cdot)$; that is

$$H(\lambda^*_{t+1}) = \int_{\underline{\lambda}}^{\bar{\lambda}^*_{t+1}} \int_{\underline{\lambda}}^{\bar{\lambda}} F(d\lambda_{t+1}; \lambda_t) H(d\lambda_t).$$

The planner thus chooses consumption and investment policies which maximize the expected present value of discounted utility. In the remainder of this section, we present an alternative interpretation of this (centrally planned) economy which is consistent with competitive equilibrium. The connections to the planned economy of problem (P) will be apparent.

Our recursive competitive economy is assumed to produce two goods—a consumption good, and an investment (capital) good. At the beginning of each period, firms observe the shock to productivity (λ_t) and purchase capital and labour from individuals at competitively determined rates. Both capital and labour are used to produce the two output goods. Individuals use their wages and the proceeds from the sale of capital to buy the consumption good (c_t) and the investment good (i_t) at the end of the period. This investment good is used as capital (k_{t+1}) available for sale to firms next period. The firm thus liquidates at the end of each period. The process continues recursively. With capital and labour competitively prices each period, the firm's objective function in this economy is especially simple—maximize period profits.

2.1. *Technology*

Firms are assumed to produce under stochastic constant returns to scale.[5] The production function relating period t output (c_t+i_t), input labour (l_t), and capital (z_t) is $l_t f(z_t/l_t)\lambda_t$ where the function $f(\quad)$ is assumed to be increasing, strictly concave, and continuous.[6] The shock (λ_t) follows a stationary Markov process with strictly positive bounded support. The firm's period t production possibility set $Y(\lambda_t)$ is thus defined by:

$$Y(\lambda_t)=\{(c_t, -l_t, -z_t, i_t)\in R^4\colon\ c_t,\ l_t,\ z_t,\ i_t\geqq 0;\ c_t+i_t\leqq l_t f(z_t/l_t)\lambda_t\}.$$

Each period the firm chooses a commodity vector $y=(c,-l,-z,i)$ from $Y(\lambda)$ which maximizes its profits.

As firms produce under constant returns to scale, the value of the firm in an Arrow–Debreu economy must be zero in a literal sense. We can, however, relate firms in our model to those observed in real capital markets. This can be done by defining the value of the capital purchased at the beginning of the period by the firm $(p_z z_t)$ to be its pre-dividend value $(VF^t_{\text{pre div}})$ and the value of the capital sold after production $(p_i i_t)$ to be the ex-dividend value of the firm $(VF^t_{\text{ex div}})$. It follows that the period dividend must be $p_z z_t - p_i i_t$.

An alternative and equivalent approach to the one developed in this paper would be to assume that the firm alone owns the capital, for which there is no market. Rather, each firm would issue one perfectly divisible equity share. These shares would be traded in a stock market and ownership of a share as of the beginning of time period t would entitle its owner to the firm's dividends (that is, the firm's output after all factors of production have been paid). This approach parallels that of Brock (1982). Since the firms in this case are operating in a competitive environment, an appropriate objective function would be to maximize the value of shares plus dividends.[7] In Prescott and Mehra (1980), Example 3, it is shown that Brock's (1982) model can be mapped into the structure developed in this paper. Thus, either approach will yield the same equilibrium allocations.

As discussed earlier, firms in our model liquidate at the end of each period, which may be counter to the reader's intuition. However, we choose this approach because it greatly simplifies the analysis. Capital, being in the household's commodity vector, is directly priced. Rather than both firms and households solving dynamic problems, only the households must do this. The problem faced by the firms is static.

2.2. *Preferences*

The infinitely lived representative individual orders his preferences over feasible consumption plans (defined below) by

$$E[\textstyle\sum_{t=0}^{\infty}\beta^t u(c_t)],\qquad 0<\beta<1,$$

where $E(\ )$ is the expectation operator, β is the period discount factor and c_t represents period t consumption. The period utility function $u(\ )$ is assumed to be strictly concave, increasing and differentiable. Capital supplied by individuals, (z_t), is constrained by available capital (k_t). Thus the individual's period t consumption possibility set may be described as

$$x(k_t) = \{(c_t, -l_t, -z_t, i_t) \in R^4 \colon c_t \geqq 0; 0 \leqq l_i \leqq 1; z_t \leqq k_t; i_t \geqq 0\}.$$

Each period the individual selects a commodity vector $x = (c, -l, -z, i)$ from $X(k)$.[8]

The individual's holding of capital at the beginning of the next sale period (k_{t+1}) equals the amount of new capital purchased this period (i_t). Thus $k_{t+1} = i_t$ specifies how capital holdings depend upon the prior period's decision.

2.3. *Equilibrium*

For this economy, all relevant information for individual decision making in period t can be characterized by the triple $(k_t, \underline{k}_t, \lambda_t)$, which we will refer to as the (period t) state of the individual. Here k_t denotes the individual's holdings of capital, while $\underline{k}_t$ denotes the distribution of capital among the other individuals in the economy. As all individuals are assumed to be identical (and thus have identical holdings), the distribution can be summarized by the holdings of a representative individual and hence $\underline{k}_t$ has the same dimensionality as k_t. (Anticipating the discussion on equilibrium, we will find that $k_t = \underline{k}_t$ in equilibrium. In order to specify correctly the consumer's problem, however, each consumer must be free to vary k_t.) The current period realization of the random shock constitutes the third component of the state. (In equilibrium, the state of the economy may thus be characterized by $(\underline{k}_t, \lambda_t)$.)

The dynamic programming techniques that we employ in this paper necessarily force us to consider only stationary equilibria. Since we are ultimately interested in comparative dynamics, however, this choice of techniques is reasonable. More general techniques (e.g. optimal control theory) would allow for the study of non-stationary equilibria.

As the structure of this economy is time invariant, economic agents solve a similar problem each period. Since we are interested in competitive equilibrium, we assume optimizing and price taking behaviour on the part of all agents. The firm's optimization problem is particularly simple in our framework. Firms face a sequence of static problems and the firm simply produces so as to maximize profits each period given market prices of capital $(p_k(\underline{k}_t, \lambda_t))$, labour $(p_l(\underline{k}_t, \lambda_t))$, consumption $(p_c(\underline{k}_t, \lambda_t))$ and investment goods $(p_i(\underline{k}_t, \lambda_t))$. Consumers maximize their expected discounted utility of consumption over feasible plans subject to their budget constraint. We assume that the economy is closed under the assumption of rational expectations (Muth (1961) and Lucas and Prescott (1971)); that is, the prices and price distributions on which the economic agents base their consumption–investment–production decisions are exactly the same as those that result as a consequence of their decisions through market clearing. Thus the vector of current prices

$$p(\underline{k}_t, \lambda_t) = (p_c(\underline{k}_t, \lambda_t), p_l(\underline{k}_t, \lambda_t), p_k(\underline{k}_t, \lambda_t), p_i(\underline{k}_t, \lambda_t))$$

and the future distribution of prices are determined *endogenously* as a function of the state of the economy.

The consumer does not have a well-defined decision problem until the equilibrium law of motion $\underline{k}_{t+1} = g(\underline{k}_t, \lambda_t)$ and the vector of prices $p(\underline{k}_t, \lambda_t)$ are specified. Knowledge of these quantities along with knowledge of the (conditional) shock distribution function

$F(\lambda_{t+1}; \lambda_t)$ is sufficient for forming predictive distributions of future prices and selecting optimal current actions. This leads to the following equilibrium definition:

Definition. A Recursive Competitive Equilibrium for this economy is characterized by the following functions:

(i) an almost everywhere continuous vector function $p(\underline{k}_t, \lambda_t)$;

(ii) an almost everywhere continuous vector function v, where $v(k_{t+1}, \underline{k}_{t+1}, \lambda_{t+1})$ values capital taken into next period k_{t+1} conditional upon the next period capital holdings of other individuals, $\underline{k}_{t+1}$, and next period's realization of the shock, λ_{t+1};

(iii) a (time stationary) period allocation policy, x, where $x(k_t, \underline{k}_t, \lambda_t)$ specifies the individual decision as a function of the current individual state;

(iv) a state contingent vector, y, where $y(\underline{k}_t, \lambda_t)$ specifies the decision of the firm;

(v) a continuous function g, where $\underline{k}_{t+1} = g(\underline{k}_t, \lambda_t)$ is the law of motion of capital stock; such that

(a) The allocation policy $x(k_t, \underline{k}_t, \lambda_t)$ maximizes

$$\left\{u(c_t) + \beta \int v(k_{t+1}, g(\underline{k}_t, \lambda_t), \lambda_{t+1}) F(d\lambda_{t+1}; \lambda_t)\right\}$$

subject to $x \in X(k_t)$ and

$$p(\underline{k}_t, \lambda_t) \cdot x_t \leqq 0.$$

(Utility maximization subject to a budget constraint.)

(b) For all $y_t \in Y(\lambda_t)$, the allocation rule maximizes valuation $y_t \cdot p(\underline{k}_t, \lambda_t)$ over all $y_t \in Y(\lambda_t)$. (Profit maximization.)

(c) $y(\underline{k}_t, \lambda_t) = x(\underline{k}_t, \underline{k}_t, \lambda_t)$. (Capital supplied equals capital demanded.)

(d) $g(\underline{k}_t, \lambda_t) = i(\underline{k}_t, \underline{k}_t, \lambda_t)$. (The law of motion of the representative consumer's capital stock is consistent with the maximizing behaviour of agents.)

(e) $v(k_t, \underline{k}_t, \lambda_t) = u(c(k_t, \underline{k}_t, \lambda_t)) + \beta \int v(i(k_t, \underline{k}_t, \lambda_t), \quad g(\underline{k}_t, \lambda_t), \lambda_{t+1}) F(d\lambda_{t+1}; \lambda_t)$. (If this condition is satisfied then the consumer is using an appropriate value function to evaluate capital stock taken into the subsequent period. As the function v satisfies the optimality equation, the representative consumer is maximizing discounted expected utility given the process generating prices and his initial capital stock holdings.)

Given this framework we have the following result:

Theorem 1. *Under the assumptions of this section a recursive competitive equilibrium exists and it is Pareto optimal.*

Proof. Prescott and Mehra (1980). ||

3. PRICING OF CAPITAL ASSETS AND DETERMINANTS OF RISK PREMIA

This section further develops our intertemporal model to allow for the pricing of capital assets and the characterization of risk premia. We also consider the issue of the economy's convergence to a unique stationary state.

3.1. *The decision problem of the representative firm*

Firm behaviour can be easily characterized in our framework. Firms face a sequence of static problems and the firm simply produces so as to maximize profits in each period, given the market price vector $(p_c(\underline{k}_t, \lambda_t), (p_l(\underline{k}_t, \lambda_t), (p_z(\underline{k}_t, \lambda_t), (p_i(\underline{k}_t, \lambda_t))$.

The firm's problem then is

$$\max p_c(\underline{k}_t, \lambda_t)c_t + p_i(\underline{k}_t, \lambda_t)i_t - p_z(\underline{k}_t, \lambda_t)z_t - p_l(\underline{k}_t, \lambda_t)l_t$$
$$\text{s.t. } c_t + i_t \leqq \lambda_t l_t f(z_t/l_t).$$

Assuming an interior solution, the first-order conditions for a maximum are

$$p_c(\underline{k}_t, \lambda_t) = p_i(\underline{k}_t, \lambda_t)$$
$$p_z(\underline{k}_t, \lambda_t) = \lambda_t p_c(\underline{k}_t, \lambda_t) f'(z_t/l_t),$$

and

$$p_l(\underline{k}_t, \lambda_t) = \lambda_t p_c(\underline{k}_t, \lambda_t)[f(z_t/l_t) - (z_t/l_t)f'(z_t/l_t)].$$

With constant returns to scale technology, the maximum profit is zero. Since aggregate consumption is equal to wages plus dividends, we have

$$p_c(\underline{k}_t, \lambda_t)c_t = \lambda_t p_c(\underline{k}_t, \lambda_t) l_t[f(z_t/l_t) - (z_t/l_t)f'(z_t/l_t)] + \lambda_t p_c(\underline{k}_t, \lambda_t) z_t f'(z_t/l_t) - p_c(\underline{k}_t, \lambda_t) i_t,$$

where the value of the dividend,

$$d_t = p_z(\underline{k}_t, \lambda_t)z_t - p_i(\underline{k}_t, \lambda_t)i_t$$

is just the pre-dividend value of the firm less its ex-dividend value.

With $p_c(\underline{k}_t, \lambda_t)$ chosen as the numeraire and knowing that in *equilibrium* (see the consumer's problem below) one unit of labour and $\underline{k}$ units of capital are supplied inelastically; i.e. $l_t \equiv 1$ and $z_t = \underline{k}_t$, these first-order conditions may be rewritten as

$$p_c(\underline{k}_t, \lambda_t) = p_i(\underline{k}_t, \lambda_t) = 1 \tag{1}$$
$$p_z(\underline{k}_t, \lambda_t) = \lambda_t f'(\underline{k}_t) \tag{2}$$
$$p_l(\underline{k}_t, \lambda_t) = \lambda_t[f(\underline{k}_t) - \underline{k}_t f'(\underline{k}_t)] \tag{3}$$
$$c_t = \lambda_t f(\underline{k}_t) - i_t \tag{4}$$
$$d_t = p_z(\underline{k}_t, \lambda_t)z_t - i_t. \tag{5}$$

3.2. *Decision problem of the representative individual*

The functional equation associated with the individual's maximization problem is

$$v(k_t, \underline{k}_t, \lambda_t) = \max_{c_t, i_t \geqq 0}\left\{u(c_t) + \beta \int v(k_{t+1}, \underline{k}_{t+1}, \lambda_{t+1})F(d\lambda_{t+1}; \lambda_t)\right\}$$
$$\text{s.t. } p_c(\underline{k}_t, \lambda_t)c_t + p_i(\underline{k}_t, \lambda_t)i_t = p_z(\underline{k}_t, \lambda_t)z_t + p_l(\underline{k}_t, \lambda_t)l_t$$
$$z_t \leqq k_t; \qquad k_{t+1} = i_t; \qquad l_t \leqq 1; \qquad \underline{k}_{t+1} = g(\underline{k}_t, \lambda_t).$$

Forming the Lagrangian we get

$$\begin{aligned} L(c_t, i_t, l_t, z_t, \theta_1, \theta_2, \theta_3) = u(c_t) + \beta \int v(k_{t+1}, \underline{k}_{t+1}, \lambda_{t+1}) F(d\lambda_{t+1}; \lambda_t) \\ + \theta_1(p_z(\underline{k}_t, \lambda_t) z_t + p_l(\underline{k}_t, \lambda_t) l_t - p_c(\underline{k}_t, \lambda_t) c_t - p_i(\underline{k}_t, \lambda_t) i_t) \\ + \theta_2(1 - l_t) + \theta_3(k_t - z_t), \end{aligned}$$

where θ_1, θ_2, and θ_3 are Lagrangian multipliers. The first order conditions (assuming an interior solution) yield

$$u'(c_t) = p_c(\underline{k}_t, \lambda_t)\theta_1$$

$$\beta \int v_1(k_{t+1}, \underline{k}_{t+1}, \lambda_{t+1}) F(d\lambda_{t+1}; \lambda_t) = p_i(\underline{k}_t, \lambda_t)\theta_1 = u'(c_t) \cdot \frac{p_i(\underline{k}_t, \lambda_t)}{p_c(\underline{k}_t, \lambda_t)}$$

$$\theta_1 p_l(\underline{k}_t, \lambda_t) - \theta_2 = 0$$

$$\theta_1 p_z(\underline{k}_t, \lambda_t) - \theta_3 = 0.$$

Since θ_1, $p_l(\underline{k}_t, \lambda_t)$, and $p_z(\underline{k}_t, \lambda_t)$ are all positive, we have $\theta_2 > 0$ and $\theta_3 > 0$. Hence $l_t = 1$ and $z_t = k_t$; that is, individuals supply one unit of labour and k_t units of capital inelastically (a fact utilized earlier in equations (1)–(5); note that in equilibrium $k_t = \underline{k}_t$).

Using the envelope theorem, and letting the price of the consumption good be the numeraire, we get

$$\beta \int u'(c_{t+1}) p_z(\underline{k}_{t+1}, \lambda_{t+1}) F(d\lambda_{t+1}; \lambda_t) = u'(c_t) p_i(\underline{k}_t, \lambda_t), \tag{6}$$

where

$$c_{t+1} = c(\underline{k}_{t+1}, \underline{k}_{t+1}, \lambda_{t+1}).$$

Equation (6) is the fundamental equation for the pricing of capital assets.[9] It equates the loss in utility associated with carrying one additional unit of capital to the discounted expected utility of the resulting additional consumption next period. To carry over one additional unit of the capital good, $p_i(\underline{k}_t, \lambda_t)$ units of the consumption good must be sacrificed and the resulting loss in utility is $p_i(\underline{k}_t, \lambda_t)u'(c_t)$. By selling this additional unit of capital next period, $p_z(\underline{k}_{t+1}, \lambda_{t+1})$ additional units of the consumption good can be consumed and

$$\beta \int u'(c_{t+1}) p_z(\underline{k}_{t+1}, \lambda_{t+1}) F(d\lambda_{t+1}; \lambda_t)$$

is the expected value of the incremental utility next period. In equilibrium, these quantities must be equated.

With these results, it is now possible to determine the equilibrium "returns and risk premia".

3.3. *Equilibrium returns and risk premia*

There are two equivalent approaches to determine the equilibrium return on the risky asset. Since there is only one risky asset, the return on this asset is also (trivially) the return on the market, which we will denote by $R_M(\cdot)$.

The first approach involves realizing that the market return during the interval t, $t+1$ is, by definition, the return realized by individuals by *purchasing* the risky asset at time t at price $p_i(\underline{k}_t, \lambda_t)$ and *selling* it at time $t+1$ at price $p_z(\underline{k}_{t+1}, \lambda_{t+1})$.

$$\therefore \quad 1+R_M(\underline{k}_t, \lambda_t, \lambda_{t+1}) = \frac{p_z(\underline{k}_{t+1}, \lambda_{t+1})}{p_i(\underline{k}_t, \lambda_t)}. \tag{7}$$

Alternatively, the same result can be obtained by using the definition that the period return by owning the firm is

$$1+R_M(\underline{k}_t, \lambda_t, \lambda_{t+1}) = \frac{VF^{t+1}_{\text{pre div}}}{VF^{t}_{\text{ex div}}} = \frac{p_z(\underline{k}_{t+1}, \lambda_{t+1})z_{t+1}}{p_i(\underline{k}_t, \lambda_t)i_t}$$

or

$$1+R_M(\underline{k}_t, \lambda_t, \lambda_{t+1}) = \frac{p_z(\underline{k}_{t+1}, \lambda_{t+1})}{p_i(\underline{k}_t, \lambda_t)},$$

since

$$z_{t+1} = i_t$$

which is identical to equation (7). Note that at time t, $R_M(\underline{k}_t, \lambda_t, \lambda_{t+1})$ is a random variable since λ_{t+1} is uncertain.

Rewriting equation (6) as

$$\beta \int \frac{u'(c_{t+1}) \cdot p_z(\underline{k}_{t+1}, \lambda_{t+1})}{u'(c_t) \cdot p_i(\underline{k}_t, \lambda_t)} F(d\lambda_{t+1}; \lambda_t) = 1, \tag{8}$$

and substituting for $p_z(\underline{k}_{t+1}, \lambda_{t+1})/p_i(\underline{k}_t, \lambda_t)$ from equation (7), we get

$$\beta \int \frac{u'(c(\underline{k}_{t+1}, \underline{k}_{t+1}, \lambda_{t+1}))}{u'(c(\underline{k}_t, \underline{k}_t, \lambda_t))}[1+R_M(\underline{k}_t, \lambda_t, \lambda_{t+1})]F(d\lambda_{t+1}; \lambda_t) = 1, \tag{9}$$

where we deliberately stress the state dependence.

Consider now the derivation of the risk free rate. We can introduce a financial instruments market where a riskless asset is traded, this asset being in zero net supply. Since we consider a single consumer economy, and since the net demand for this asset must be zero in equilibrium (it being in zero net supply), its existence does not affect the equilibrium as long as its price is obtained in the usual way from the first order conditions of the (single) consumer. Let $dp_{R_F}(\underline{k}_t, \lambda_t)$ be the price at time t of a security which pays one unit of consumption next period when the state of nature is between λ_{t+1} and $\lambda_{t+1} + d\lambda_{t+1}$; then in equilibrium

$$dp_{R_F}(\underline{k}_t, \lambda_t)u'(c(\underline{k}_t, \underline{k}_t, \lambda_t)) = \beta u'(c(\underline{k}_{t+1}, \underline{k}_{t+1}, \lambda_{t+1})) \operatorname{Prob}[\lambda \in \lambda_{t+1} + d\lambda_{t+1}, \lambda_{t+1}|\lambda_t].$$

Given the current state $(\underline{k}_t, \lambda_t)$, the price of a security that pays a unit of consumption in every state next period is therefore

$$p_{R_F}(\underline{k}_t, \lambda_t) = \beta \int \frac{u'(c(\underline{k}_{t+1}, \underline{k}_{t+1}, \lambda_{t+1}))}{u'(c_t(\underline{k}_t, \underline{k}_t, \lambda_t))} F(d\lambda_{t+1}; \lambda_t). \tag{10}$$

Defining $R_F(\underline{k}_t, \lambda_t)$ as the period risk free rate (conditional on the current state), we have, by definition,

$$1+R_F(\underline{k}_t, \lambda_t)=\frac{p_c(\underline{k}_{t+1}, \lambda_{t+1})}{p_{R_F}(\underline{k}_t, \lambda_t)}=\frac{1}{p_{R_F}(\underline{k}_t, \lambda_t)}$$

(since $p_c(\underline{k}_{t+1}, \lambda_{t+1})=1$) and thus

$$\frac{1}{1+R_F(\underline{k}_t, \lambda_t)}=\beta \int \frac{u'(c_{t+1})}{u'(c_t)} F(d\lambda_{t+1}; \lambda_t), \tag{11}$$

where we again suppress the state dependence of consumption.

All the information necessary to compute the aggregate risk premium is now complete. Expanding equation (9) yields:

$$\left[\beta \int \frac{u'(c_{t+1})}{u'(c_t)} F(d\lambda_{t+1}; \lambda_t)\right] \cdot \left[\int (1+R_M(\underline{k}_t, \lambda_t, \lambda_{t+1}))F(d\lambda_{t+1}; \lambda_t)\right]$$
$$+\beta \operatorname{cov}\left[\frac{u'(c_{t+1})}{u'(c_t)}, R_M(\underline{k}_t, \lambda_t, \lambda_{t+1})|\underline{k}_t, \lambda_t\right]=1.$$

Substituting from equation (11) and integrating with respect to the conditional distribution of λ_{t+1}, equation (12) obtains:

$$\frac{1+\bar{R}_M(\underline{k}_t, \lambda_t)}{1+R_F(\underline{k}_t, \lambda_t)}=1-\beta \operatorname{cov}\left[\frac{u'(c_{t+1})}{u'(c_t)}, R_M(\underline{k}_t, \lambda_t, \lambda_{t+1})|\underline{k}_t, \lambda_t\right], \tag{12}$$

where

$$\bar{R}_M(\underline{k}_t, \lambda_t)=\int R_M(\underline{k}_t, \lambda_t, \lambda_{t+1})F(d\lambda_{t+1}; \lambda_t)$$

is the expected market return conditional on the current state. Now define the expected period risk premium $\bar{\pi}(\underline{k}_t, \lambda_t)$, conditioned on the current state as

$$\bar{\pi}(\underline{k}_t, \lambda_t)=\bar{R}_M(\underline{k}_t, \lambda_t)-R_F(\underline{k}_t, \lambda_t).$$

Rearranging and simplifying equation (12) gives the desired result:

$$\bar{\pi}(\underline{k}_t, \lambda_t)=\beta(1+R_F(\underline{k}_t, \lambda_t)) \operatorname{cov}\left[-\frac{u'(c_{t+1})}{u'(c_t)}, R_M(\underline{k}_t, \lambda_t, \lambda_{t+1})|\underline{k}_t, \lambda_t\right]. \tag{13}$$

Making use of equations (7), (1) and (2), this may be rewritten as

$$\bar{\pi}(\underline{k}_t, \lambda_t)=\beta(1+R_F(\underline{k}_t, \lambda_t)) \operatorname{cov}\left[-\frac{u'(c_{t+1})}{u'(c_t)}, \lambda_{t+1}f'(\underline{k}_{t+1})|\underline{k}_t, \lambda_t\right]. \tag{14}$$

Equation (14) is an expression for the (conditional) expected risk premium and states that this premium depends upon the covariance of the marginal rate of substitution of consumption and the marginal productivity of capital per unit labour. This covariability is what results in risk. If technology were riskless or if individuals were risk neutral the covariance term in (14) would be zero and there would be no systematic risk. (With risk neutral individuals there would still be risk in the sense of technological uncertainty, but there would be no systematic risk.) Systematic risk and hence risk premia thus depend upon preferences and technology, in particular upon their covariability as discussed above.

We emphasize again that the expressions for the risk free rate, the expected return on the market and the expected risk premium depend upon the current state. Specifically the covariance term in (14) is a conditional covariance and in general will vary through time as the state variables change. This is also true of the covariance term in Breeden's (1979) model hence the "beta" in that model will in general be non-stationary, a fact that should be taken into account when testing it.[10] Most equilibrium models in finance have failed to emphasize the distinction between expected returns conditional on the current state (as in Merton (1973) and Breeden (1979)) and expected returns calculated with respect to the stationary distribution of state variables (see the discussion below). We feel that in most applications it would seem more appropriate to focus on these expectations as being "averaged" across all possible capital stock-shock states; i.e. unconditional state independent expectations.

Such a calculation would only be feasible, however, if the Markov process on capital stock (and thus also the Markov processes on consumption and output), defined by $\underline{k}_{t+1} = i(\underline{k}_t, \underline{k}_t, \lambda_t)$ converges to a well-defined unique stationary state. Conditions sufficient for this to be so have been investigated in Donaldson and Mehra (1983) and are given below; in effect they require a minor further specialization of the economy described in Section 2, in particular, a specialization of the shock transition function $F(\lambda_{t+1}; \lambda_t)$:

Assumption 1. The production technology $f(\ )$ is increasing, bounded, strictly concave, and three times continuously differentiable, with $f(0) = 0$, and $f'(0) = \infty$.

Assumption 2. The period utility function $u(\ )$ is increasing, bounded, strictly concave, and three times continuously differentiable and displays constant or decreasing relative risk aversion. Furthermore, for any interval (ε, M), where $\varepsilon > 0, M < \sup_k f(k)$, there is a W such that

$$\infty > W > \frac{u'''(\gamma_1)}{|u''(\gamma_2)|}, \quad \forall \gamma_1, \gamma_2 \text{ in } (\varepsilon, M).$$

Assumption 3. The shock λ is subject to a stationary Markov process with strictly positive bounded support $[\underline{\lambda}, \bar{\lambda}]$, where $\underline{\lambda} > 0$. Furthermore, we require that $F(\cdot\,; \lambda')$ stochastically dominate $F(\cdot\,; \lambda)$ in the first degree (Rothschild and Stiglitz (1971)) whenever $\lambda' > \lambda$. (This requirement is intended to capture the notion that tomorrow is more likely to be similar to today than radically different.) Lastly, for all λ_1, λ_2 in the support of $F(\cdot\,;\cdot)$,

$$\int |F(d\lambda; \lambda_2) - F(d\lambda; \lambda_1)| < B|\lambda_2 - \lambda_1|,$$

for

$$B = \frac{\frac{1}{2} f(k_L)\underline{\lambda} M}{\beta \lambda M_1 M_2}.$$

The terms M, M_1, M_2 are defined respectively, by $\sup_{c_L \leqq c \leqq c_U} -(u''(c)), \beta \max[f'(k_0), f'(k_L/2)]$ and $u'(c_L/2)$, where $[c_L, c_U]$ and $[k_L, k_U]$ are intervals which bound the asymptotic consumption and capital paths. Since β, $\underline{\lambda}$, and $\bar{\lambda}$ alone determine these bounds, B is independent of $F(\cdot\,;\cdot)$.[11] (This requires that if two production shocks today do not differ by very much, the conditional distributions of next period's shock must be closely similar.)

Theorem 2. *Consider the economy of Section 2 as satisfying Assumptions 1, 2 and 3. Then the optimal aggregate investment and consumption functions exist and are stationary, continuous, and monotone functions of output. Furthermore, these functions define Markov processes on capital stock, consumption, and output which converge in measure to unique stationary probability distributions.*

Proof. See Donaldson and Mehra (1983). ||

Theorem 2 (which summarizes the results of Donaldson and Mehra (1983)) may be viewed as an application of the results in Futia (1982), who studies the convergence of general Markov processes in the context of the associated Markov operators. In the model of this paper, capital stock and consumption are invariant functions of output. Thus convergence of the Markov process on output is sufficient for the convergence of the capital stock and consumption processes as well. Section IV(i) of Donaldson and Mehra (1983) guarantees that the Markov operator associated with the process on output is stable; Section IV(ii) guarantees that it is tight (by demonstrating that the ergodic set is a compact connected interval). Thus by Futia (1981), Theorem 2.9, we know an invariant output measure exists. Section IV(iii) of Donaldson and Mehra (1983) further demonstrates that the output process is irreducible, thus satisfying Futia's (1983) Uniqueness Criterion 2.11. Hence the invariant measure is unique (Futia (1983), Theorem 2.12). It may not, however, mirror the long run average behaviour of the process (a property which is needed to legitimize our simulation exercise which follows in Section 4). But as the kernel of the output process has a continuous density, we may apply Theorem 4.6 of Futia (1983) to conclude that the Markov operator associated with the output process is quasi-compact. This is sufficient (Futia 1983, Theorem 2.10) to guarantee that the invariant measure does indeed mirror the process' long run average behaviour.

Let $(K, G(k))$ and $(\Lambda, H(\lambda))$ denote, respectively, the range and stationary probability distribution for the Markov processes on capital stock and the random shock and let $Q(K, \lambda)$ denote the joint distribution. The average (across all states) risk free rate, market return, and market risk premium are thus:

$$\bar{R}_F = \int_{k\in K} \int_{\lambda\in\Lambda} R_F(k, \lambda) Q(dK, d\lambda) \tag{15}$$

where $R_F(k, \lambda)$ is defined by equation (11);

$$\bar{\bar{R}}_M = \int_{k\varepsilon K} \int_{\lambda\varepsilon\Lambda} \bar{R}_M(k, \lambda) Q(dK, d\lambda) \tag{16}$$

where $\bar{R}_M(k, \lambda)$ is found by integrating $R_M(k_t, \lambda_t, \lambda_{t+1})$ (as defined in equation (7)) with respect to the conditional distribution of λ_{t+1}, given λ_t; and

$$\bar{\bar{\pi}} = \int_{k\varepsilon K} \int_{\lambda\varepsilon\Lambda} [\bar{R}_M(k, \lambda) - R_F(k, \lambda)] Q(dK, d\lambda). \tag{17}$$

Under the interpretation embodied by equations (7)–(14) it is clear that the risk premium (indeed, all quantities) varies as the state variables vary. For the interpretation embodied in equations (15)–(17), these quantities are constant. These Assumptions 1–3 may be viewed as sufficient conditions for ensuring that the "beta" of this model will be stationary; consequently, time series estimation would be fully appropriate.

4. COMPARATIVE DYNAMICS

In this section we numerically compute the solution functions to problem (P) (see Section 2) for various parameter choices and use these functions then to compute the associated time averaged risk free rate, market rate, and risk premium. In this way we are able to gain first insight into the effect of parameter choices on these quantities.

In particular, we explicitly solve problem (P) for the cases in which $u(c)=(c^{\gamma}-1)/\gamma$, $\gamma\in\{-2,-1,0(\ln c),0\cdot25,0\cdot5\}$, $\beta\in\{0\cdot8,0\cdot9,0\cdot95\}$, $f(k)=Ak^{\alpha}$, $A=2/3$, $\alpha\in\{0\cdot24,0\cdot5\}$, $\lambda_t\in\{1\cdot5,\ 1,\ 0\cdot5\}$, and for which the transition density for $\tilde{\lambda}_t$ is approximated as a 3×3 transition probability matrix $\{\Phi\}$, where the ϕ_{uv}-th entry indicates the probability that $\lambda_{t+1}=\lambda_v$ given $\lambda_t=\lambda_u$. These choices allow us to study the influence of changes in individual risk aversion (γ) and time preference (β). To understand the effects of shock correlation we further solve problem (P) for a variety of symmetric matrices $\{\Phi\}$ where $\phi_{uu}=c$, $\phi_{uv}=(1-c)/2$, $u\neq v$. By allowing ϕ_{uu} to assume values from the set $\{0\cdot333,0\cdot5,0\cdot7,0\cdot9\}$, we are able to model progressively greater shock correlation.

Our parameter choices (in particular the fact that $A=2/3$) constrain the problem in such a way as to ensure that the range of each of the series of consumption, output, and capital stock lies in the interval $[0,1]$. Choosing capital stock as the state variable, we define a partition $\Gamma=\{k_i\}$ of this range, for which, for any two adjacent elements of the partition k_i, k_{i+1}, $k_{i+1}-k_i=0\cdot001$. This partition then serves as the set of feasible states for capital stock. To determine the optimal investment and consumption policies for each of the parameter choices, we follow the customary dynamic programming method of seeking a fixed point to the related functional equation by a sequence of approximating iteration. Under this procedure, the optimal investment policies $i(k_i,\lambda_u)=(k_{t+1})$ are found as limits of sequences $i_n(k_i,\lambda_u)$, where $i_n=i_n(\cdot,\cdot)$ solves:

$$V^n(k_i,\lambda_u)=\max\{u(f(k_i)\lambda_u-i_n)+\beta\textstyle\sum_{n=1}^{3}V^{n-1}(i_n,\lambda_v)\phi_{uv}\},$$
$$i_n\in\Gamma,\qquad i_n\leqq f(k_i)\lambda_u.$$

Given these investment functions, the transition probability matrix for the Markov process on capital stock can then be constructed. Given this matrix, the stationary distribution on capital stock (from which can be derived the analogous distribution on output and consumption) can be obtained by solving the associated linear programming problem. Following a computational routine designed to reflect equations (9)–(17), we then computed the time averaged risk free rate, market rate, and risk premium. These results are presented in Tables I and II below:

As would be expected, in all cases the aggregate risk premium rises as agents become more risk averse (γ more negative). Similarly, the risk free rate also declines with greater risk aversion. This latter fact is easily explained by noting that, with greater risk aversion, agents would be willing to pay more and more today for the same certain payments tomorrow—thereby lowering the rate of return on the implied security.

We also notice that R_M first rises and then declines with greater risk aversion. The increase in R_M has a simple interpretation: as agents become more risk averse, we would expect them to require greater average returns. Less apparent is the eventual decline in the market returns. As is discussed in Danthine and Donaldson (1981) and Danthine *et al.* (1983), the reduced variation in consumption (corresponding to the decrease in γ) can be realized only at the expense of greater variation in output. This increase in variation results from the enormously expanded range of the stationary capital stock distribution (for $\phi_{uu}=0\cdot33$ and $\beta=0\cdot95$, the range of the stationary distribution on capital stock for $\gamma=0\cdot5$ and $\gamma=-2$ are, respectively, $[0\cdot05,0\cdot11]$ and $[0\cdot02,0\cdot35]$). Since

TABLE I

$\beta=0{\cdot}95,\ \alpha=0{\cdot}25$

	$\gamma=0{\cdot}5$	$\gamma=0$	$\gamma=-1$	$\gamma=-2$
$\phi_{uu}=1/3$	$\bar{\bar{R}}_M=0{\cdot}164$	$\bar{\bar{R}}_M=0{\cdot}235$	$\bar{\bar{R}}_M=0{\cdot}269$	$\bar{\bar{R}}_M=0{\cdot}208$
	$\bar{R}_F=0{\cdot}043$	$\bar{R}_F=0{\cdot}010$	$\bar{R}_F=-0{\cdot}103$	$\bar{R}_F=-0{\cdot}226$
	$\bar{\bar{\pi}}=0{\cdot}121$	$\bar{\bar{\pi}}=0{\cdot}225$	$\bar{\bar{\pi}}=0{\cdot}371$	$\bar{\bar{\pi}}=0{\cdot}434$
$\phi_{uu}=0{\cdot}5$	$\bar{\bar{R}}_M=0{\cdot}144$	$\bar{\bar{R}}_M=0{\cdot}199$	$\bar{\bar{R}}_M=0{\cdot}206$	$\bar{\bar{R}}_M=0{\cdot}123$
	$\bar{R}_F=0{\cdot}034$	$\bar{R}_F=-0{\cdot}011$	$\bar{R}_F=-0{\cdot}139$	$\bar{R}_F=-0{\cdot}268$
	$\bar{\bar{\pi}}=0{\cdot}110$	$\bar{\bar{\pi}}=0{\cdot}210$	$\bar{\bar{\pi}}=0{\cdot}345$	$\bar{\bar{\pi}}=0{\cdot}391$
$\phi_{uu}=0{\cdot}7$	$\bar{\bar{R}}_M=0{\cdot}113$	$\bar{\bar{R}}_M=0{\cdot}147$	$\bar{\bar{R}}_M=0{\cdot}115$	$\bar{\bar{R}}_M=0{\cdot}022$
	$\bar{R}_F=0{\cdot}035$	$\bar{R}_F=-0{\cdot}009$	$\bar{R}_F=-0{\cdot}148$	$\bar{R}_F=-0{\cdot}283$
	$\bar{\bar{\pi}}=0{\cdot}078$	$\bar{\bar{\pi}}=0{\cdot}156$	$\bar{\bar{\pi}}=0{\cdot}262$	$\bar{\bar{\pi}}=0{\cdot}306$
$\phi_{uu}=0{\cdot}9$	$\bar{\bar{R}}_M=0{\cdot}076$	$\bar{\bar{R}}_M=0{\cdot}088$	$\bar{\bar{R}}_M=0{\cdot}062$	$\bar{\bar{R}}_M=-0{\cdot}014$
	$\bar{R}_F=0{\cdot}046$	$\bar{R}_F=0{\cdot}024$	$\bar{R}_F=-0{\cdot}071$	$\bar{R}_F=-0{\cdot}200$
	$\bar{\bar{\pi}}=0{\cdot}030$	$\bar{\bar{\pi}}=0{\cdot}064$	$\bar{\bar{\pi}}=0{\cdot}133$	$\bar{\bar{\pi}}=0{\cdot}186$

TABLE II

$\beta=0{\cdot}99,\ \alpha=0{\cdot}25$

	$\gamma=0{\cdot}5$	$\gamma=0$	$\gamma=-1$	$\gamma=-2$
$\phi_{uu}=1/3$	$\bar{\bar{R}}_M=0{\cdot}116$	$\bar{\bar{R}}_M=0{\cdot}185$	$\bar{\bar{R}}_M=0{\cdot}219$	$\bar{\bar{R}}_M=0{\cdot}169$
	$\bar{R}_F=0{\cdot}000$	$\bar{R}_F=-0{\cdot}031$	$\bar{R}_F=-0{\cdot}136$	$\bar{R}_F=-0{\cdot}248$
	$\bar{\bar{\pi}}=0{\cdot}116$	$\bar{\bar{\pi}}=0{\cdot}216$	$\bar{\bar{\pi}}=0{\cdot}355$	$\bar{\bar{\pi}}=0{\cdot}417$
$\phi_{uu}=0{\cdot}5$	$\bar{\bar{R}}_M=0{\cdot}099$	$\bar{\bar{R}}_M=0{\cdot}151$	$\bar{\bar{R}}_M=0{\cdot}147$	$\bar{\bar{R}}_M=0{\cdot}070$
	$\bar{R}_F=-0{\cdot}008$	$\bar{R}_F=-0{\cdot}051$	$\bar{R}_F=-0{\cdot}179$	$\bar{R}_F=0{\cdot}301$
	$\bar{\bar{\pi}}=0{\cdot}107$	$\bar{\bar{\pi}}=0{\cdot}201$	$\bar{\bar{\pi}}=0{\cdot}326$	$\bar{\bar{\pi}}=0{\cdot}371$
$\phi_{uu}=0{\cdot}7$	$\bar{\bar{R}}_M=0{\cdot}073$	$\bar{\bar{R}}_M=0{\cdot}103$	$\bar{\bar{R}}_M=0{\cdot}086$	$\bar{\bar{R}}_M=-0{\cdot}012$
	$\bar{R}_F=-0{\cdot}005$	$\bar{R}_F=-0{\cdot}047$	$\bar{R}_F=-0{\cdot}173$	$\bar{R}_F=-0{\cdot}304$
	$\bar{\bar{\pi}}=0{\cdot}078$	$\bar{\bar{\pi}}=0{\cdot}150$	$\bar{\bar{\pi}}=0{\cdot}259$	$\bar{\bar{\pi}}=0{\cdot}292$
$\phi_{uu}=0{\cdot}9$	$\bar{\bar{R}}_M=0{\cdot}032$	$\bar{\bar{R}}_M=0{\cdot}045$	$\bar{\bar{R}}_M=0{\cdot}027$	$\bar{\bar{R}}_M=-0{\cdot}058$
	$\bar{R}_F=0{\cdot}003$	$\bar{R}_F=-0{\cdot}016$	$\bar{R}_F=-0{\cdot}099$	$\bar{R}_F=-0{\cdot}237$
	$\bar{\bar{\pi}}=0{\cdot}029$	$\bar{\bar{\pi}}=0{\cdot}061$	$\bar{\bar{\pi}}=0{\cdot}126$	$\bar{\bar{\pi}}=0{\cdot}179$

the market return is the marginal product of capital, and since greater risk aversion forces the economy to operate at higher capital stock levels, average market returns must ultimately be reduced. That returns are occasionally reported as negative follows from the nature of our shock structure (a bad realization ($\lambda=1/2$) following a good realization ($\lambda=3/2$) can cut output more than in half, for example), and rounding errors resulting from the coarseness of our capital stock partition.

We note also that as persistence increases (ϕ_{uu} rises) the aggregate risk premium declines. This is not unexpected as greater persistence reduces the period of period uncertainty faced by agents and thus the essential period riskiness of their investment decisions. This decline is observed despite the fact that increased persistence can be shown to increase the variance of the stationary distributions of all three—consumption, investment, and output—series (see, once again, Danthine *et al.* (1983) for an elaboration).

Lastly, we observe that an increase in β effects a decline in the risk free and market rates for all parameter choices. In this model context it is well known (see Danthine and Donaldson (1981)) that an increase in β will cause individuals to invest a greater proportion of output for all output levels. This has the consequence of shifting the stationary distribution of capital stock to a range of higher capital stock levels. Since market returns reflect marginal productivities of capital, it is natural that a decline in the average market return be observed. In a similar vein, an increase in β, by causing agents to value a certain consumption unit next period even more highly, will induce them to be willing to "pay" even more today for the claim to such certain consumption. Here the risk free rate of return is observed to fall.

The shock variation of our model ($\lambda = 1/2, 1, 3/2$) was chosen for illustrative purposes and indeed the variation is larger than can reasonably be expected in an actual economy.[12] Thus it would not be meaningful to compare the risk premia computed in this section to that observed in reality.[13] These computations show qualitatively, however, that our model provides results consistent with our economic intuition. In particular, risk premia are observed to decline with reduced uncertainty. We feel therefore the usefulness of the model lies in its ability to make qualitative predictions about the intertemporal pricing of capital assets as a function of the changes in the economic environment.

A legitimate empirical test of the model would be to estimate the Markov process defining the time path of actual GNP, and then construct a shock matrix which would yield similar output behaviour. The derived $\bar{R}_F$, $\bar{\bar{R}}_M$, and $\bar{\bar{\pi}}$ implied by this specification could then be compared with the above empirical data. We leave this—somewhat involved—empirical issue to a later note.[14]

Our model fails to mirror reality in at least one other way. Here the shares of income going to capital and to labour are equally risky (it is impossible to shift risk from capitalist to labourers for they are the same group of individuals). In practice, however, labour's share is much less risky than the share going to capital. This is because labour contracts are in a way similar to insurance contracts; labour's claim on output is largely fixed and is negotiated prior to the actual realization of the output.[15] Hence, in reality, more uncertainty in output is absorbed by the owners of capital than labour. To this extent, our model underestimates the actual risk premium.

5. CONCLUDING REMARKS

This paper develops a simple general equilibrium model, characterized by both informational and allocative efficiency, in which asset prices are linked to the underlying technology and preferences. The framework presented enables one to analyse the qualitative effects of shifts in productivity, tastes, time preferences, changes in uncertainty in the economic environment, and alternative tax policies on the intertemporal behaviour of asset prices.

Although the one capital good setting of our model appears restrictive, this feature was chosen only for ease of computation. In fact, as mentioned earlier, the appropriate equilibrium representation of the J-good analogue can be found in Prescott and Mehra (1980). The derivations of this paper could be extended to that context in a natural way; however, the actual numerical computation of approximate risk premia for a wide class of parameters would become considerably more involved.

The description of the economy as one with homogeneous consumers is also restrictive. Nevertheless, the current state of equilibrium theory requires this if the parametric sensitivity of equilibrium is to be studied.

Not unexpectedly, the model demonstrates quite explicitly that an individual's desire to accept risk—and thus the returns that must be paid to encourage risk taking—depend upon real consumption levels. Such results emphasize the partial equilibrium nature of typical asset pricing models, such as the familiar CAPM.

Our model also points out the distinction between the time averaged risk free and market rates and risk premium and the analogous period (conditional on the state) values.

When theoretical financial models speak of the risk free and market rates, which of these concepts, in fact, is being used? For one period models, only the (conditional) period concept would seem appropriate. Yet the distinction is rarely observed.

Adaptations of our model can be used to study the relative pricing of many financial assets (the J-good analogue) and the influence of monetary growth on the pricing of financial assets. We examine these issues in subsequent papers.

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NOTES

1. A representative, though not an exhaustive list includes the work of Black (1972), Constantinides (1980), Fama (1977), Merton (1973), Ross (1976), Rubinstein (1974 and 1976), Stapleton and Subrahmanyam (1978). The Roas (1976) paper suggests an alternative to the equilibrium approach. It is an asset pricing model based on arbitrage considerations. For an excellent survey, see LeRoy (1979).

2. In addition to Prescott and Mehra (1980), Brock (1982) has developed a general equilibrium model with production that characterizes equilibrium in financial markets. While Brock's model is of the same tradition as ours, he is primarily concerned with deriving a theoretical foundation for the Ross (1976) arbitrage pricing theory.

Cox, Ingersoll and Ross (1980) have also developed an intertemporal asset pricing model, though in continuous time. They are interested, however, in the term structure of interest rates, an issue that we do not consider. While they do not offer a general existence proof, Huang (1982) suggests that this is possible.

3. There has been some confusion in the finance literature as to the meaning of (informational) efficiency. For a clarification, see Lucas (1978), especially p. 1429. Prices in our economy are a function of the underlying state variables, but they are not a sufficient statistic for these variables. We merely wish to emphasize that "informational efficiency" as commonly used in the finance literature, is little more than asserting a rational expectations equilibrium.

4. The analyses can easily be extended to price $J>1$ types of capital goods. For a definition of equilibrium in this setting see Example 3 in Prescott and Mehra (1980). In this case, the discussion in Section 3 below will have to be modified. In particular, there will be equations similar to (6) and (7) for each of the J capital types. The computations in Section 4 would also be considerably increased.

5. The constant returns to scale assumption is innocuous. When this assumption is not satisfied, it is typically because some factor such as land is owned rather than rented by the firm and thus is not included in the commodity vector. In general, a factor can be added to the commodity vector such that the resulting technology set displays constant returns to scale. See McKenzie (1959) for details.

6. To clarify our notation further, k_t denotes capital held by individuals and available for sale to firms in period t, while z_t denotes the capital firms actually choose to purchase in period t. In equilibrium, of course, $z_t = k_t$. Furthermore the firms know λ_t at the time they make their production decisions.

7. For a discussion as to the appropriateness of value maximization see Hart (1979).

8. We use the symbols (c, l, z, i) to characterize both the commodity points for the firm and the consumers. However, as discussed in the section on equilibrium below, the commodity point of the firm is a function of (k, λ) and that of the consumer $(k, \underline{k}, \lambda)$. To clarify, the c in the commodity point of the firm is a function specifying the consumption good supplied by the firm; perhaps a more correct notation would be $c^s(\underline{k}, \lambda)$. Similarly, the c in the commodity point of the individual is the amount of the consumption good demanded by the individual and should technically be written as $c^d(k, \underline{k}, \lambda)$. In equilibrium of course $c^s = c^d$. Since there is little room for confusion, we stick to the simplified notation in this paper.

9. Notice that this is similar to the first order conditions of Lucas (1978) and Brock (1979).

10. It has been pointed out to us by Fischer Black that Cornell (1981) has made this observation independently. See also Constantinides (1980). For a complete discussion of the relevant empirical issues, see Hansen *et al.* (1981).

11. For Assumption 3 to be non vacuous we need to ensure that B is not zero. The latter is ensured if $k_L > 0$. The issue of the positivity of k_L is exhaustively considered in Mirman and Zilcha (1976), who show that it is possible for k_L to be zero. However, as they also point out (Mirman and Zilcha, 1976, p. 127), the difficulty in securing an explicit example in which this is observed suggests that the phenomena is largely pathalogical. In this light we will assume that $k_L > 0$ and hence B is not zero.

12. Computational tractability was another reason for choosing this shock structure—more pronounced shock variability enables us to incorporate a less fine partition Γ while still sensing the change in returns resulting from different capital stock and shock levels. By using a fairly coarse partition, as we do, the ultimate computation of the stationary state (matrix inversion) becomes more manageable.

13. Ibbotson and Sinquefield report a risk premium of 8·7% based on historical data for the period 1926–1978.

14. See Mehra and Prescott (1983) for such a test in a pure exchange economy.

15. This was brought to our attention by Edward Prescott.

REFERENCES

BHATTACHARYA, S. (1981), "Notes on Multiperiod Valuation and the Pricing of Options", *Journal of Finance*, **36**, 163–180.

BLACK, F. (1972), "Capital Market Equilibrium with Restricted Borrowing", *Journal of Business*, **45**, 444–455.

BREEDEN, D. T. (1979), "An Intertemporal Asset Pricing Model with Stochastic Investment Opportunities", *Journal of Financial Economics*, **7**, 265–296.

BROCK, W. A. (1979), "An Integration of Stochastic Growth Theory and the Theory of Finance, Part 1: The Growth Model", in Green, J. and Scheinkman, J. (eds.) *General Equilibrium, Growth & Trade* (New York: Academic Press).

BROCK, W. A. (1982), "Asset Prices in a Production Economy", in McCall, J. J. (ed.) *The Economics of Information and Uncertainty* (Chicago: University of Chicago Press).

CONSTANTINIDES, G. M. (1980), "Admissible Uncertainty in the Intertemporal Asset Pricing Model", *Journal of Financial Economics*, **8**, 71–86.

CONSTANTINIDES, G. M. (1983), "Intertemporal Asset Pricing with Heterogeneous Consumers and without Demand Aggregation", *Journal of Business* (forthcoming).

CORNELL, B. (1981), "The Consumption Based Asset Pricing Model; A Note on Potential Tests and Applications", *Journal of Financial Economics*, **9**, 103–108.

COX, J., INGERSOLL, J. and ROSS, S. (1983), "A Theory of the Term Structure of Interest Rates", *Econometrica* (forthcoming).

DANTHINE, J. P. and DONALDSON, J. B. (1981), "Stochastic Properties of Fast vs. Slow Growing Economies", *Econometrica*, **49**, 1007–1033.

DANTHINE, J. P., DONALDSON, J. B. and MEHRA, R. (1983), "On the Impact of Shock Persistence on the Dynamics of a Recursive Economy", *European Economic Review*, **22**, 147–166.

DONALDSON, J. B. and MEHRA, R. (1983), "Stochastic Growth with Correlated Production Shocks", *Journal of Economic Theory*, **29**, 282–312.

FAMA, E. F. (1970), "Efficient Capital Markets: A Review of Theory and Empirical Work", *Journal of Finance*, **25**, 383–417.

FAMA, E. F. (1977), "Risk Adjusted Discount Rates and Capital Budgeting under Uncertainty", *Journal of Financial Economics*, **5**, 3–24.

FUTIA, C. A. (1982), "Invariant Distributions and the Limiting Behavior of Markovian Economic Models", *Econometrica*, **50**, 377–408.

GROSSMAN, S. J. and STIGLITZ, J. (1976), "On Stockholder Unanimity in Making Production and Financial Decisions" (Technical Report No. 224, IMSS, Stanford University).

HANSEN, L., RICHARD, S. and SINGLETON, K. (1981), "Econometric Implications of the Intertemporal Capital Asset Pricing Model" (Working Paper, GSIA, Carnegie-Mellon University).

HART, O. D. (1975), "On the Optimality of Equilibrium when the Market Structure is Incomplete", *Journal of Economic Theory*, **11**, 418–443.

HART, O. D. (1979), "On Shareholder Unanimity in Large Stock Market Economies", *Econometrica*, **47**, 1057–1083.

HUANG, C. (1982), "Information Structure and Equilibrium Asset Prices" (Discussion Paper, Graduate School of Business, Stanford University).

IBBOTSON, R. G. and SINQUEFIELD, R. Q. (1979) *Stocks, Bonds, Bills, and Inflation: Historical Returns* (1926–1978) (Financial Analysts Research Foundation).

KANODIA, C. (1980), "Effects of Stockholder Information on Corporate Decisions and Capital Market Equilibrium", *Econometrica*, **48**.

KREPS, D. M. (1979), "Three Essays on Capital Markets" (Research Paper No. 499, Graduate School of Business, Stanford University).

LeROY, S. F. (1979), "Securities Prices under Risk Uncertainty and Near Risk-Neutrality" (University of Chicago Working Paper).

LeROY, S. F. (1982), "Expectations' Model of Asset Prices: A Survey of Theory", *Journal of Finance*, **37**, 185–217.

LINTNER, J. (1965), "The Valuation of Risky Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets", *Review of Economics and Statistics*, **47**, 13–37.

LUCAS, R. E., Jr. (1978), "Asset Prices in an Exchange Economy", *Econometrica*, **46**, 1426–1446.

LUCAS, R. E., Jr. and PRESCOTT, E. C. (1971), "Investment under Uncertainty", *Econometrica*, **39**, 659–681.

McKENZIE, L. W. (1959), "On the Existence of General Equilibrium for a Competitive Market", *Econometrica*, **27**, 54–71.

MEHRA, R. and PRESCOTT, E. C. (1983), "The Equity Premium: A Puzzle" (Working Paper, Columbia University).

MERTON, R. (1973), "An Intertemporal Asset Pricing Model", *Econometrica*, **41**, 867–887.

MILNE, F. (1979), "Consumer Preferences, Linear Demand Functions and Aggregation in Competitive Asset Markets", *Review of Economic Studies*, **46**, 407–418.

MIRMAN, L. J. and ZILCHA, I. (1976), "Unbounded Shadow Prices for Optimal Stochastic Growth Models", *International Economic Review*, **17**, 121–132.

MOSSIN, J. (1966), "Equilibrium in a Capital Asset Market", *Econometrica*, **34**, 768–783.

MUTH, J. F. (1961), "Rational Expectations and the Theory of Price Movements", *Econometrica*, **29**, 315–335.

PRESCOTT, E. C. and MEHRA, R. (1980), "Recursive Competitive Equilibrium: The Case of Homogeneous Households", *Econometrica*, **48**, 1365–1379.

RADNER, R. (1974), "A Note on Unanimity of Stockholders' Preferences Among Alternative Production Plans: A Reformulation of the Ekern–Wilson Model", *The Bell Journal of Economics and Management Science*, **5**, 181–186.

RADNER, R. (1972), "Existence of Equilibrium of Plans, Prices and Price Expectations in a Sequence of Markets", *Econometrica*, **40**, 289–303.

ROLL, R. (1970) *The Behavior of Interest Rates* (New York: Basic Books).

ROSS, S. A. (1976), "The Arbitrage Theory of Capital Asset Pricing", *Journal of Economic Theory*, **13**, 341–349.

ROTHSCHILD, M. and STIGLITZ, J. E. (1971), "Increasing Risk II: Its Economic Consequences", *Journal of Economic Theory*, **4**, 66–84.

RUBINSTEIN, M. (1974), "An Aggregation Theorem for Securities Markets", *Journal of Financial Economics*, **1**, 225–244.

RUBINSTEIN, M. (1976), "The Valuation of Uncertain Income Streams and the Pricing of Options", *Bell Journal of Economics*, **7**, 407–425.

SHARPE, W. F. (1964), "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk", *Journal of Finance*, **19**, 425–442.

STAPLETON, R. C. and SUBRAHMANYAM, M. G. (1978), "A Multiperiod Equilibrium Asset Pricing Model", *Econometrica*, **46**, 1077–1096.