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International Economic Review, Vol. 29, No. 1 (Feb., 1988), 131-135.

Stable URL:
http://links.jstor.org/sici?sici=0020-6598%28198802%2929%3A1%3C131%3AOTEARO%3E2.0.CO%3B2-T

International Economic Review is currently published by Economics Department of the University of Pennsylvania.

INTERNATIONAL ECONOMIC REVIEW
Vol. 29, No. 1, February 1988

ON THE EXISTENCE AND REPRESENTATION OF EQUILIBRIUM IN AN ECONOMY WITH GROWTH AND NONSTATIONARY CONSUMPTION*

BY RAJNISH MEHRA[1]

1. INTRODUCTION

The paper by Lucas (1978) "Asset Prices in an Exchange Economy" initiated a paradigmatic change in the Theory of Equilibrium Pricing of Risky Assets. It was followed by several papers including those by Brock (1979, 1982), Prescott and Mehra (1980) and Donaldson and Mehra (1984) who generalize Lucas' model to a production setting.

In this paper, we examine a variation of Lucas' pure exchange model. In Lucas' model the level of consumption follows a Markov process. Observing the large increases in per capital consumption in the past, we postulate that the growth rate of consumption follows a Markov process, an assumption that enables us to capture the nonstationarity in the consumption series. Our extension is particularly relevant for empiricists interested in testing consumption–based asset pricing models in the Lucas-Prescott research tradition.

The introduction of nonstationarity into the consumption process is a nontrivial exercise necessitating an extension of competitive equilibrium theory, which we discuss and develop in Section 3 below. We establish the existence and representation of a competitive equilibrium of the Debreu variety for an important class of economies. We show the equilibrium has a valuation system that can be represented as a dot product.

Economies of the type we consider here have received a lot of attention, especially in the literature on asset pricing. However, no one to our knowledge has offered a proof of existence of equilibrium or has established a representation of the dot product type. The important papers by Hansen and Singleton (1982, 1983) typifies the approach. Existence and representation are simply assumed, and the authors directly invoke the necessary first order conditions for optimality. Brock (1979, 1982), on the other hand, does not connect with the Debreu approach.

Since we deal with growth and nonstationarities in the consumption process, the analysis becomes nonstandard. In the absence of growth, one could simply apply the theorem in Lucas (1978). The crux of the analysis is to identify a commodity point and space with the associated topology for which the utility

* Manuscript received February 1985; revised January 1987.
[1] I am especially indebted to Edward C. Prescott for his perceptive insights and his generosity in sharing them. Financial support from the National Science Foundation is gratefully acknowledged.

functions have the necessary continuity properties and the production sets have nonempty interiors so that the conditions of Debreu (1954) can be verified.

This paper consists of four sections. Section 2 describes the economy. Section 3 considers the existence and optimality of equilibrium. Section 4 concludes the paper.

2. THE ECONOMY

The economy we consider was judiciously selected so that the joint process governing the growth rates in aggregate per capita consumption and asset prices would be stationary and easily determined. The economy has a single representative "stand-in" household. This unit orders its preferences over random consumption paths by

$$E_0\left\{\sum_{t=0}^{\infty}\beta^t U(c_t)\right\} \tag{1}$$

where c_t is per capita consumption, β is the subjective time discount factor, $E_0\{\cdot\}$ is the expectation operator conditional upon information available at time zero (which denotes the present time) and U: $R_+ \to R$ is the increasing concave utility function. To insure that the equilibrium return process is stationary, the utility function is further restricted to be of the constant relative risk aversion class,

$$U(c, \alpha) = \frac{c^{1-\alpha} - 1}{1 - \alpha}; \qquad 0 < \alpha < \infty. \tag{2}$$

The parameter α measures the curvature of the utility function. When $\alpha = 1$, the utility function is defined to be the logarithmic function, the limit of the above function as α approaches one.

We assume that there is one productive unit producing the perishable consumption good and there is one equity share that is competitively traded. Since only one productive unit is considered, the return on this share of equity is also the return on the market. The firm's output is constrained to be less than or equal to y_t. It is the firm's dividend payment in period t as well.

The growth rate in y_t is subject to a Markov chain; that is:

$$y_{t+1} = x_{t+1} y_t \tag{3}$$

where $x_{t+1} \in \{\lambda_1, \ldots, \lambda_n\}$ and

$$\Pr\{x_{t+1} = \lambda_j;\ x_t = \lambda_i\} = \phi_{ij}. \tag{4}$$

It is also assumed that the Markov chain is ergodic. The λ_i are all nonnegative and $y_0 > 0$. The random variable y_t is observed at the beginning of the period, at which time dividend payments are made. All securities are traded ex-dividend. We also assume that the matrix A with elements $a_{ij} \equiv \beta\phi_{ij}\lambda_j^{1-\circ}$ for $i, j = 1, \ldots, n$ is stable; that is, $\lim A^m$ as $m \to \infty$ is zero. In the Appendix, it is shown that this is necessary and sufficient for the expected utility to exist if the stand-in household consumes y_t every period.

3. EQUILIBRIUM

In order for this to be a Debreu (1954) competitive equilibrium model, it is necessary to map our model into his structure. This requires, among other things, a specification of a linear space, L, to serve as the commodity space. Given that in our economy, economic activity takes place over an infinite number of periods, the space is necessarily infinite dimensional. Our commodity space is the normed linear space of infinite sequences of vectors with the tth vector indexed by the event, $e_t \equiv (x_1, \ldots, x_t)$. The set of possible period t events, E_t, is finite having cardinality n^t. The norm for $z \in L$ is

$$\| z \| = \sup_t \max_{e_t \in E_t} \left| \frac{z_t(e_t)}{y_t} \right|$$

where $y_t = y_0 x_1 \ldots x_t$ is the event contingent maximum output of the firm. The element $z_t(e_t)$ is the quantity of the good delivered in period t conditional upon e_t occurring.

The households consumption set is

$$C = \{c \in L: \quad y_t/2 \leq c_t(e_t) \quad \text{all} \quad e_t \in E_t \quad \text{all} \quad t\},$$

which is stronger than the requirement that consumption be non-negative. This helps in establishing the continuity of the preference ordering on C induced by (1).

The endowment of the stand-in household is the zero element of L and the firm's production possibility set is

$$W = \{w \in L: \quad w_t(e_t) \leq y_t \quad \text{all} \quad e_t \in E_t \quad \text{all} \quad t\}.$$

This completes the representation of our economy in the Debreu framework.

The allocation $c_t^*(e_t) = w_t^*(e_t) = y_t$ is a Pareto optimum as more is preferred to less. In the Appendix, the expected utility of plan c^* is shown to exist. As plan $2c^*$ belongs to C, the element c^* is not a saturation point for the stand-in household. The appendix also establishes the continuity of the utility function.

The consumption possibility set C is convex; the expected utility functional u: $C \to R$ is concave and continuous; the production possibility set W is convex and has an interior point; c^* is not a saturation point for the stand-in household. By Theorem 2 of Debreu (1954, page 590), this optimum can be supported by a valuation equilibrium subject to the conditions of the Remark (page 591). The conditions of the Remark are satisfied for a point that exists in C having valuation less than c^*. The point with $c_0 = y_0/2$ and $c_t(e_t) = c_t^*(e_t)/2$ is such a point.

This theorem of Debreu does not guarantee that the equilibrium valuation function v: $L \to R$ has the dot product representation, which is required in the subsequent analysis. The needed result is now established.[2]

Let L^n be the linear subspace of L for which $z_t = 0$ for $t > n$. Let $\pi_n(z)$ denote

[2] This result could also be established by verifying Mackey continuity of preferences and then applying a theorem of Bewley (1970, 1972) or Brown and Lewis (1981).

the projection of z on L^n. The following valuation function p, which does have a dot product representation, will be shown to also support the optimum allocation:

$$p(z) = \lim_{n\to\infty} v[\pi_n(z)] = \sum_t \sum_{e_t} p_t(e_t) z_t(e_t).$$

If $z \in C$ implied $\pi_n(z) \in C$, the result would be an application of Theorem 1 in Prescott and Lucas (1972, page 418). Their theorem holds under the following slightly weaker conditions. Letting c^n denote the element with $c_t^n(e_t) = c(e_t)$ for $t \leq n$ and $c_t^n(e_t) = y_t$ for $t > n$, the Prescott-Lucas assumptions that $c \in C$ implies $\pi_n(c) \in C$ and that $c, c' \in C$ and $u(c) > u(c')$ implies $u[\pi_n(c)] > u(c')$ for sufficiently large n are modified by replacing $\pi_n(c)$ by c^n. This slightly more general version of their theorem is established by substituting $p(\cdot) + \lim v(0^n)$ for $p(\cdot)$ wherever it appears in their proof, where 0 denotes the zero element of L. [Note that in this setting $v(c^n) = v(\pi_n(c)) + v(0^n)$. Hence, $\lim v(c^n) = p(c) + \lim v(0^n)$]

4. CONCLUSIONS

In this paper, we model the nonstationarity of consumption series associated with the large increases in past per capita consumption by examining a variant of Lucas' Asset Pricing model. We define an equilibrium and demonstrate its existence and optimality. This facilitates the development of a framework for testing a class of consumption-based pricing models, with a view to evaluating their predictions against the yardstick of actual observation. We pursue the empirical implications of these models in Mehra and Prescott (1985).

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APPENDIX

We first establish that the expected utility of the element $c^* \in C$ with the $c_t^*(e_t) = y_t$ exists. Let $v_t(y, i)$ be the expected utility for the first $t + 1$ periods of the plan if $y_0 = y$ and $x_0 = \lambda_i$. It satisfies the recursion

$$v_{t+1}(y, i) = \frac{y^{1-\alpha} - 1}{1 - \alpha} + \beta \sum_j \phi_{ij} v_t(\lambda_j y, j). \tag{A1}$$

for $t = 0, 1, \ldots$. The initializing function is

$$v_0(y, i) = \frac{y^{(1-\alpha)} - 1}{1 - \alpha}. \tag{A2}$$

By definition, the expected utility of c^* is the limit of $v_t(y, i)$ as t goes to infinity.

It is easily verified by mathematical induction that

$$v_t(y, i) = \frac{\gamma_{it}}{(1 - \alpha)} y^{(1-\alpha)} - \frac{1 - \beta^{t+1}}{(1 - \alpha)(1 - \beta)} \tag{A3}$$

by noting it is true for $t = 0$ and using (A1) to conclude that if it is true for t then it is true for $t + 1$. The substitution of (A3) into (A1) yields

$$\gamma_{i,t+1} = 1 + \beta \sum_j \phi_{ij} \lambda_j^{(1-\alpha)} \gamma_{jt} \qquad i = 1, \ldots, n. \tag{A4}$$

The requirement for the expected utility to exist is that the difference equation (A4) converge given $\gamma_{i0} = 1$ for all i. It will occur if and only if the $n \times n$ matrix $A = [\beta \phi_{ij} \lambda_j^{(1-\alpha)}]$ has eigenvalues which all lie within the unit circle in the complex plan or equivalently that $\lim A^n = 0$. This is true by assumption.

The expected utility exists for all $c \in C$ and is continuous because $c \in C$ constraints event contingent consumption $c_t(e_t)$ to be at least half $c_t^*(e_t)$ and not more than $\| c \|$ times $c_t^*(e_t)$. This uniformly bounds the percentage difference between the $c_t(e_t)$ insuring the expected utility of c exists given the expected utility of c^* exists. Continuity follows because the sequence $c_n \in L$ converging to $c \in L$ requires the percentage difference between $c_n(e_t)$ and $c(e_t)$ go to zero uniformly in t and e_t. This implies the limit of $u(c_n)$ is $u(\lim c_n)$.

REFERENCES

BEWLEY, T., "Equilibrium Theory with an Infinite-Dimension Commodity," Ph.D. dissertation, Berkeley, 1970.

BEWLEY, T., "Existence of Equilibria in Economics with Infinitely Many Commodities," *Journal of Economic Theory* 4 (1972), 514–540.

BROCK, W. A., "An Integration of Stochastic Growth Theory and the Theory of Finance, Part 1: The Growth Model," in J. Green and J. Scheinkman, eds., *General Equilibrium, Growth & Trade* (New York: Academic Press, 1979).

BROCK, W. A., "Asset Prices in a Production Economy," in J. J. McCall, ed., *The Economics of Information and Uncertainty* (Chicago: University of Chicago Press, 1982).

BROWN, D. J., AND L. M. LEWIS, "Myopic Economic Agents," *Econometrica* 49 (March 1981), 359–368.

DEBREU, G., "Valuation Equilibrium and Pareto Optimum," *Proceedings of the National Academy of Sciences* 70 (1954), 588–592.

DONALDSON, J. B., AND R. MEHRA, "Comparative Dynamics of an Equilibrium, Intertemporal Asset Pricing Model," *The Review of Economic Studies* 51 (1984), 491–508.

HANSEN, L. P. AND K. J. SINGLETON, "Generalized Instrumental Variable Estimation of Nonlinear Rational Expectations Models," *Econometrica* 50 (1982), 1269–86.

——— AND ———, "Stochastic Consumption, Risk Aversion, and the Temporal Behavior of Asset Returns," *Journal of Political Economy* 91 (1983), 249–265.

LUCAS, R. E., JR., "Asset Prices in an Exchange Economy," *Econometrica* 46 (November 1978), 1429–1445.

MEHRA, R. AND E. C. PRESCOTT, "The Equity Premium: A Puzzle," *Journal of Monetary Economics* 15 (March 1985), 145–161.

PRESCOTT, E. C., AND R. E. LUCAS, JR., "A Note on Price Systems in Infinite Dimension Space," *International Economic Review* 13 (1972), 416–422.

———, AND R. MEHRA, "Recursive Competitive Equilibrium: The Case of Homogeneous Households," *Econometrica* 48 (September 1980), 1365–1379.