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RECURSIVE COMPETITIVE EQUILIBRIUM: THE CASE OF HOMOGENEOUS HOUSEHOLDS

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Recursive equilibrium theory is extended and generalized. Optimality of equilibria and supportability of optima are established in a direct way. Four economic applications are reformulated as recursive competitive equilibria and analyzed.

1. INTRODUCTION

ONE WAY OF MODELLING dynamic uncertain economic phenomena is to use Arrow-Debreu general equilibrium structures and to search for optimal actions, conditional on the sequence of realizations of all past and present random variables or shocks. An alternative approach which is proving very useful in developing testable theories is to replace the attempt to locate equilibrium sequences of contingency functions with the search for equilibrium decision rules. These decision rules specify current action as a function of a limited number of "state variables" which summarize the effects of past decisions and current information. These equilibrium decision rules must be time invariant in order to apply standard time series methods and this necessitates a recursive structure. The purpose of this paper is to further the development of recursive competitive theory in the hope of facilitating its use in economics. In particular, we expect it to prove useful in explaining regularities in asset price movements and in the co-movements of economic aggregates, both of which have been the subject of extensive empirical research.

Although our structure is a generalization of the one used by Lucas [**12**] in his model of capital asset pricing, our method of analysis is very different. As in his analysis, a homogeneous class of individuals is assumed; but, unlike his analysis, ours permits capital accumulation. This paper also subsumes the structure considered in Lucas and Prescott's [**13**] analysis of equilibrium investment under uncertainty. Optimality of recursive equilibria and supportability of Pareto optima by recursive equilibria is established in a simpler and more direct way which does not rely upon the equivalence of recursive and state-contingent equilibrium allocations.

The homogeneity of consumers is admittedly not a realistic assumption for most applications. If all heterogeneous agents discount at the same rate and conditions are satisfied that insure Pareto optimality of a competitive equilibrium, the equilibrium processes for economic aggregates and prices will be *observationally equivalent* to those for some homogeneous consumer economy. Heterogeneity, of course, will typically necessitate the introduction of supplementary securities to allocate risk.

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Recently some other equilibrium analyses of capital asset pricing have appeared. Included is the work of Brock [**4**] who exploits variational rather than recursive methods. In Section 8, we show how his structure can be formulated as a recursive equilibrium problem. In addition we present Sargent's [**18**] model in which he examines the consistency between Tobin's q theory and the equilibrium stochastic growth model of Mirman and Zilcha [**16**]. Other closely related papers are the analyses of Breeden [**3**], Constantinides [**5**], Cox, Ingersol, and Ross [**6**], Danthine [**7**], Johnsen [**9**], Kanodia [**10**], and Merton [**15**].

2. THE ECONOMY

Individuals are infinitely lived and have identical preferences and endowments of initial capital stocks. Each industry consists of many identical (except for size) firms producing capital and consumption goods. Over time each industry is subject to shocks which affect that industry's production possibility set. Since a main thrust of the analysis is the pricing of capital assets, we structure the trading as follows: individuals sell their labor services and capital stocks to firms at competitively determined prices. Firms in each industry use capital and labor to produce consumption and/or capital goods (for use next period) which they sell to the consumer. Capital goods may be industry specific; i.e., capital of type j may be used only by industry j. This formulation differs from the more conventional approach in which individuals rent capital to firms. We choose this alternative approach because it prices capital types each period which should facilitate the development of a general equilibrium theory to explain observed asset pricing regularities.

Each period, a typical consumer chooses a finite dimensional real vector x_t, his *period t commodity point*, from his *period consumption possibility* set $X(k_t)$. The vector x_t describes the quantities of commodities he actually consumes (positive) and the quantities of inputs (negative) he provides. We find it convenient to split x_t into three sets of components, $x_t = \{x_{1t}, x_{2t}, x_{3t}\}$. The components x_{2t} are all negative and correspond to the capital types supplied by the individual, while x_{1t} and x_{3t} correspond to the individual's period t consumption and the capital stock he owns at the end of period t and can sell to firms in period $t+1$.

The individual is constrained to choose $x \in X$ in each period. This set is closed and convex and imposes constraints upon components of x_1 including constraints such as labor supplied does not exceed available time. In addition the capital supply, x_2, is constrained by capital purchased last period, k. Thus, the *period consumption possibility set* given k is

$$X(k) = \{x : x \in X \text{ and } -k \leqslant x_2 \leqslant 0\}.$$

The individual's holding of capital at the beginning of the next sale period (that is k_{t+1}) equals the amount of new capital purchased this period (that is x_{3t}). Thus $k_{t+1} = x_{3t}$ specifies how capital holdings depend upon the previous period decision. The set of possible capital stocks is the positive orthant.

It is assumed that the vector of random shocks $\lambda \in R^n$ is subject to a stationary Markov process with a bounded ergodic set Λ. The transition function of the process is $F:\Lambda\times\Lambda\to[0,1]$. $F(\cdot\mid\cdot)$ is continuous in both its arguments and for fixed λ, $F(\cdot\,|\lambda)$ is the probability distribution for next period's shock.

Preferences

The representative individual orders his preferences over random consumption paths for which $x_t \in X(k_t)$ with probability one and $k_{t+1}=x_{3t}$ for all t by

$$E\left\{\sum_{t=0}^{\infty}\beta^t u(x_{1t},\lambda_t)\right\}$$

where $E\{\cdot\}$ is the expectation operator, β is the discount factor $0<\beta<1$, λ_t is a random shock vector, and $u(\cdot)$ is the period utility function. The function $u:X\times\Lambda\to R$ is assumed to be strictly increasing, strictly concave in x, and differentiable.

A measure space $(I, \mathscr{B}(I), \mu)$ of consumers is assumed, where I is the unit interval, $\mathscr{B}(I)$ the Borel subsets, and μ the Lebesque measure. To obtain the aggregate demand function, the demands of individuals are integrated rather than summed as would be done if there were a finite number of consumers. This insures consumers are small and justifies the price taking assumption. Since all individuals are alike and the measure of the unit interval is one, the density of the representative individual's demand just equals aggregate consumer demand.

Technology

Firms produce under constant returns to scale. Under constant returns to scale, the price taking industry will behave as a single price taking firm. This assumption is consistent with percentage growth rates being uncorrelated with firm size (Gibrat's Law) and the observed variability of firm size. There are a number of interesting models that satisfy this assumption including those of Lucas [**12**], Lucas and Prescott [**13**], and Sargent [**18**].

The constant returns to scale assumption is innocuous. Typically when the convex constraint set is not a cone, it is because some factor such as land is owned rather than rented by the firm and is not included in the commodity vector. In general a factor can be added to the commodity vector such that the resulting technology set for a firm is a convex cone. (See McKenzie [**14**].) The equilibrium price of this factor is the value of the firm. This is done for Brock's and Lucas' capital asset models (see Section 8).

Each period a firm in industry j chooses a commodity point y_{jt} from its *production possibility set* $Y_j(\lambda_t)$, where $Y_j(\lambda_t)$ is a closed convex set. The commodity vector y_{jt} is an element of the same finite-dimensional commodity space as x_t. The constant returns to scale assumption implies that given a random shock λ and a positive γ, if $y_i \in Y_j(\lambda)$ then $\gamma y_j \in Y_j(\lambda)$.

Let $Y(\lambda)$ be the aggregate production set obtained by summing sets $Y_j(\lambda)$ over $j \in J$. The correspondence $Y(\lambda)$ is assumed continuous. We also assume that there is a bounded set K such that $k \in K$, $\lambda \in \Lambda$ and $y \in X(k) \cap Y(\lambda)$ implies $y_3 \in K$. This assumption is satisfied for growth models as there is a maximal sustainable capital stock. It will be satisfied given the constant returns to scale assumption if some factor, say labor, is essential for production and its supply is bounded. (See Arrow and Hahn [**1**, Chapter 3].) The technology $Y(\lambda)$ is assumed available to all.

3. SINGLE AGENT RECURSIVE STATISTICAL DECISIONS THEORY

A particularly useful class of dynamic structures are those of the recursive or time invariant variety, since the resulting equilibrium is a system of time invariant stochastic differential equations as assumed in most econometric testing. The structure considered in this paper falls into this category. In this section we present a brief description of some key concepts and results which are used in subsequent sections.

The *stationary statistical decision problem*, which was analyzed by Blackwell [**2**] and for the convex case by Lucas and Prescott [**13**], is one for which:

(i) The return function $u: X \times S \to R$ is time separable jointly in the period t decision variable $x_t \in X$ and an appropriately defined state variable $s_t \in S$. Returns are discounted by a factor $\beta (0 < \beta < 1)$ and the return function has the form

$$\sum_{t=0}^{\infty} \beta^t u(x_t, s_t). \tag{3.1}$$

These state variables include those elements which determine the technology available to the agents, such as the stock of capital goods, and those elements which specify the effect of past decisions on contemporary preferences. The state variables also include those elements which specify the relevant aspects of the agents' information sets. We emphasize that the state variables should be of minimal dimension, indexing only those factors which can (potentially) change over time. Our use of the term state differs from Arrow's use of the term. Our definition is the appropriate generalization to dynamic competitive environments of the state variable concept used by Bellman and the statistical decision theorists.

(ii) The state variable is observed or is an invertible function of observables at time t. The conditional distribution of s_{t+1} given current and past decisions depends only upon (x_t, s_t) and the conditional distribution function $F(s_{t+1}|x_t, s_t)$ is time invariant or stationary. We assume a set S exists such that

$$S = \{s' | \mathrm{pr}[s' \in S | x, s] = 1 \text{ for all } x \in X(s), s \in S\}.$$

(The prime indicates next period's value.)

(iii) The decision x_t is constrained to belong to the set $X(s_t)$. We let

$$X = \{x : x \in X(s) \text{ for some } s \in S\}.$$

A *recursive or stationary allocation policy* $\delta : S \to X$ is a measurable decision rule specifying the current decision as a function of the current state:

$$x_t = \delta(s_t). \tag{3.2}$$

It is feasible if $\delta(s_t) \in X(s_t)$ for all possible s_t.

With some continuity assumptions that are typically satisfied for economic applications there exists a unique, bounded, measurable, in fact continuous function satisfying Bellman's optimality equation:

$$v(s) = \max \{u(x, s) + \beta \int v(s')\, dF(s'|x, s)\} \tag{3.3}$$

subject to $x \in X(s)$.

Sufficient conditions for the above result are: (i) The function $u : X \times S \to R$ is continuous and bounded; (ii) the probability distribution function $F : S \times X(s) \times S \to R$ is continuous with respect to $(x, s) \in X(s) \times S$ for all $s' \in S$; (iii) the correspondence $X(s)$ is continuous almost everywhere; and (iv) the sets $X(s)$, $s \in S$, are compact. The function $v : S \to R$, under these conditions is the maximal obtainable expected returns over all feasible policies. Further, a recursive policy exists which is optimal in that its return is v. A necessary and sufficient condition for a feasible recursive policy to be optimal is that

$$v(s) = u[\delta(s), s] + \beta \int v(s')\, dF[s'|\delta(s), s] \tag{3.4}$$

where v is the unique solution to the optimality equation (3.3) for all $s \in S$.

In economic applications return functions u are typically concave and the constraint sets convex. If u is concave and the set

$$\{(x, s) : x \in X(s), s \in S\} \tag{3.5}$$

convex, then the optimal return function v is concave. If so, the maximization in (3.3) is a standard finite dimensional concave programming problem.

4. RECURSIVE COMPETITIVE EQUILIBRIUM

In Section 3 we defined the concept of a state variable. In the economy considered in this paper, all relevant information for individual decision making can be characterized by a triple $(k, \underset{\sim}{k}, \lambda)$, which we will refer to as the state of the individual. Here k is the individual's holding of capital types. The element $\underset{\sim}{k}$ is the distribution of capital types among the other individuals in the economy. As all individuals are assumed to be identical (and thus have identical holdings), the distribution can be summarized by the holdings of a representative individual and hence $\underset{\sim}{k}$ has the same dimensionality as k. (Anticipating the discussion on equilibrium we will find that $k = \underset{\sim}{k}$ in equilibrium. In order to solve correctly the consumer's problem, however, each consumer must be free to vary k.) The current period realization of the random shock λ constitutes the third component of the state. The state of the economy is characterized by $(\underline{k}, \lambda)$.

We observe that the structure of the economy is time invariant and economic agents solve a similar problem each period. A feasible *recursive or stationary allocation* can be defined as a measurable decision rule $x(k, \underline{k}, \lambda)$, specifying the period commodity point chosen by a consumer with capital k when the state of the economy is $(\underline{k}, \lambda)$ and a set of measurable decision rules $y_j(\underline{k}, \lambda)$, which specify the period commodity point chosen by each firm $j \in J$ as a function of the state of the economy such that:

(i) $\quad x(k, \underline{k}, \lambda) \in X(k), \quad$ all $\underline{k} \in K$,

(ii) $\quad y_j(\underline{k}, \lambda) \in Y_j(\lambda), \quad$ all $j \in J \quad$ and $\quad \underline{k} \in K$,

(iii) $$\sum_{j \in J} y_j(\underline{k}, \lambda) = \int_I x(\underline{k}, \underline{k}, \lambda)\, d\mu = x(\underline{k}, \underline{k}, \lambda).$$

(Since we are dealing with a homogeneous class of individuals, I is the unit interval and μ the Lebesgue measure.)

For our economy homogeneous individuals, the Pareto optima of interest are ones for which the weights for all individuals' utility functions are equal. To characterize these optima (it will turn out that there is only one), we define the equal weight aggregate utility function as follows:

$$U(\underline{x}, \lambda) = \max \int_I u(x_i, \lambda)\, d\mu$$

subject to $\underline{x} \geqslant \int_I x_i\, d\mu$ and $x_i \in X$ all i. Strict concavity of u implies the best x_i are all equal. Thus an allocation is Pareto optimal if it maximizes

$$E\left\{ \sum_{t=0}^{\infty} \beta^t u(\underline{x}, \lambda) \right\}$$

subject to $\underline{x} \in Y(\lambda) \cap X(\underline{k})$ and $\underline{k}_{t+1} = \underline{x}_{3t}$. The concavity of u, the convexity of the production possibility set Y and the consumption possibility set $X(\underline{k})$, together with the fact that the set

$$\{(\underline{x}, \underline{k}) \colon \underline{x} \in X(\underline{k}) \cap Y(\lambda) \text{ and } \underline{k} \in K\}$$

is convex and compact, ensures the existence of an optimal allocation policy.

Before proceeding to a discussion of an equilibrium in this economy we need to make some additional assumptions. We assume optimizing and price-taking behavior on the part of all agents. Firms maximize profits each period (the maximum profits will be zero because of the constant-returns-to-scale assumption) and consumers maximize their expected discounted utility of consumption over feasible plans subject to their budget constraint. We assume that the economy is closed under the assumption of rational expectations (Muth [**17**], and Lucas and Prescott [**13**]). That is, the prices and price distributions on which the economic agents base their consumption-investment-production decisions are exactly the same as those that result as a consequence of their decisions through market clearing. Thus, current prices $p(\underline{k}\lambda)$ and future distribution of prices are determined *endogenously* as a function of the state of the economy.

The structure used considerably simplifies the firm's problem. Since in each period there is a *market value* for end-of-period capital stock, the firm faces a sequence of static problems and the firm simply produces so as to maximize profits each period given market prices.

The consumer does not have a well defined decision problem, until the equilibrium law of motion, $\underline{k}'=f(\underline{k},\lambda)$, and the pricing function, $p(\underline{k},\lambda)$, are specified. Knowledge of these elements along with $F(\lambda'|\lambda)$ is sufficient for forecasting future prices (actually forming predictive probability distributions of future prices) and selection of optimal current actions. This leads us to the following.

DEFINITION—*Recursive Competitive Equilibrium*: A period equilibrium for this economy is characterized by the following functions:

(1) an almost everywhere continuous pricing function $p:K\times \Lambda\rightarrow L$, where the linear space L has the same finite dimensionality as the period commodity point;

(2) an almost everywhere continuous value function $v:K\times K\times\Lambda\rightarrow R$, where $v(k',\underline{k}',\lambda')$ values capital taken into next period, k', conditional upon the next period capital of other households, $\underline{k}'$, and the next period realization of the shock, λ';

(3) a period allocation policy $x:K\times K\times\Lambda\rightarrow R$, where $x(k,\underline{k},\lambda)$ specifies the individual decision as a function of the current individual state;

(4) a J-tuplet of economy state contingent points $\{y_j(\underline{k},\lambda)\}_{j=1}^{J}$, where $y_j:K\times\Lambda\rightarrow L$ specifies the decisions of firm $j\in J$;

(5) a continuous function $f:K\times\Lambda\rightarrow K$, where $\underline{k}'=f(\underline{k},\lambda)$ is the law of motion of capital stock;

such that for all $(k,\underline{k},\lambda)\in K\times K\times\Lambda$:

(i) Allocation policy $x(k,\underline{k},\lambda)$ maximizes $u(x_1,\lambda)+\beta\int v(x_3,f(\underline{k},\lambda),\lambda')\,dF(\lambda'|\lambda)$ subject to $x\in X(k)$ and $p(\underline{k},\lambda)\cdot x\leqslant 0$. (Utility maximization subject to a budget constraint.)

(ii) For all $y_j\in Y_j(\lambda)$, allocation rule $y_j(\underline{k},\lambda)$ maximizes valuation $y_j\cdot p(\underline{k},\lambda)$ over all $y_j\in Y_j(\lambda)$. (Profit maximization.)

(iii) $\Sigma_{j\in J}y_j(\underline{k},\lambda)=\int_I x(\underline{k},\underline{k},\lambda)\,d\mu=x(\underline{k},\underline{k},\lambda)$. (Supply equals demand.)

(iv) $f(\underline{k},\lambda)=x_3(\underline{k},\underline{k},\lambda)$. (The law of motion of the representative consumers capital stock is consistent with the maximizing behavior of agents.)

(v) $v(k,\underline{k},\lambda)=u(x_1(k,\underline{k},\lambda),\lambda)+\beta\int v(x_3(k,\underline{k},\lambda),f(\underline{k},\lambda),\lambda')\,dF(\lambda'|\lambda)$. (If this condition is satisfied, then by results cited in Section 3, the consumer is using an appropriate value function to evaluate capital stock taken into the subsequent period. As the function v satisfies the optimality equation, the representative consumer is maximizing discounted expected utility given the process generating prices and his initial capital stock holdings.)

5. PARETO OPTIMUM ALLOCATION

The equal weight Pareto optimum allocation maximizes the expected value of

$$\sum_{t=0}^{\infty} \beta^t u(\underline{x}_{1t}, \lambda_t) \tag{5.1}$$

subject to the constraints $\underline{x}_t \in X(\underline{k}_t) \cap Y(\lambda_t)$ and $\underline{k}_{t+1} = \underline{x}_{3t}$ for all t. As the function u is strictly concave, all individuals $i \in I$ consume the same commodity bundle (except for sets of measure zero). By the results listed in Section 3, the optimal social return $w^0(\underline{k}, \lambda)$ is continuous and concave in $\underline{k}$ and is the unique bounded solution to the optimality equation:

$$w(\underline{k}, \lambda) = \max \left\{ u(\underline{x}_1, \lambda) + \beta \int w(\underline{x}_3, \lambda')\, dF(\lambda'|\lambda) \right\} \tag{5.2}$$

$$\text{subject to} \quad \underline{x} \in X(\underline{k}) \cap Y(\lambda).$$

As w^0 is concave and continuous in $\underline{k}$, it is differentiable with respect to $\underline{k}$ almost everywhere. Furthermore, it is increasing in $\underline{k}$, strictly increasing in $\underline{k}$ if u is strictly increasing, since increases in $\underline{k}$ increase the set $X(\underline{k})$. The equal weight Pareto optimum allocation is denoted by $\underline{x}_t = \delta^0(\underline{k}_t, \lambda_t)$.

The concavity, with respect to $\underline{x}$, of the function

$$h(\underline{x}, \lambda, w^0) \equiv u(\underline{x}_1, \lambda) + \beta \int w^0(\underline{x}_3, \lambda')\, dF(\lambda'|\lambda), \tag{5.3}$$

the convexity of the closed set $Y(\lambda)$, and the convexity of the set $X(\underline{k})$ implies the existence of a separating hyperplane. This hyperplane separates the set of points belonging to $X(\underline{k})$ and yielding a higher value for $h(\underline{x}, \lambda, w^0)$ than obtained by $\underline{x} = \delta^0(\underline{k}, \lambda)$, from the production possibility set $Y(\lambda)$. In order to insure the existence of an almost everywhere continuous pricing function selection we make the following assumption:

ASSUMPTION: *If $\underline{k}_A > \underline{k}_B$ (i.e. strict inequality for at least one component), then for any point $\underline{x}_B \in X(\underline{k}_B) \cap Y(\lambda)$ there exists a point $\underline{x}_A \in X(\underline{k}_A) \cap Y(\lambda)$ such that $\underline{x}_A > \underline{x}_B$ and $\underline{x}_{1A} > \underline{x}_{1B}$.*

This assumption implies that with more capital more current consumption of some good is possible without any reduction in capital available for next period. Given $u(\underline{x}_1, \lambda)$ is assumed strictly increasing, strictly concave, and differentiable, the function h is differentiable with respect to $\underline{x}$. Consequently, the hyperplane $p^0(\underline{k}, \lambda)$ which is tangent to the upper contour set at $x = \delta^0(\underline{k}, \lambda)$ is unique. The continuity of the constraint correspondences $X(\underline{k})$ and $Y(\lambda)$ and of the function h are sufficient to insure that the pricing function $p^0(\underline{k}, \lambda)$ is continuous.

The pricing vector conditional upon $(\underline{k}, \lambda)$ has the same finite dimensionality as the period commodity point $\underline{x}$ and is indexed by the economy state $(\underline{k}, \lambda)$. This implies that

$$p^0(\underline{k}, \lambda) \cdot y \leq p^0(\underline{k}, \lambda) \cdot \delta^0(\underline{k}, \lambda) = 0$$

for all $y \in Y(\lambda) = \Sigma Y_j(\lambda)$. The zero profit arises because of constant returns to scale. In addition, provided there is an $\underset{\sim}{x} \in X(k)$ such that $p^0(\underset{\sim}{k}, \lambda) \cdot \underset{\sim}{x} < p^0(\underset{\sim}{k}, \lambda) \cdot \delta^0(\underset{\sim}{k}, \lambda)$, it also implies that $h(\delta^0(\underset{\sim}{k}, \lambda), \lambda, w^0) \geq h(\underset{\sim}{x}, \lambda, w^0)$ for all $x \in X(k)$ such that

$$p^0(\underset{\sim}{k}, \lambda) \cdot \underset{\sim}{x} \leq p^0(\underset{\sim}{k}, \lambda) \cdot \delta^0(\underset{\sim}{k}, \lambda) = 0;$$

that is, it implies utility maximization subject to the budget constraint. This pricing function along with the law of motion $\underset{\sim}{k}_{t+1} = \delta_3^0(\underset{\sim}{k}_t, \lambda_t)$, a continuous function, will be used to support a recursive competitive equilibrium.

6. OPTIMALITY OF RECURSIVE EQUILIBRIUM

An equilibrium is an optimum if the representative individual's utility $v(\underset{\sim}{k}, \underset{\sim}{k}, \lambda) = w^0(\underset{\sim}{k}, \lambda)$.

THEOREM: *A recursive equilibrium is optimal under the assumptions of Sections 2 and 3.*

PROOF: The nature of the proof is to establish that $v(\underset{\sim}{k}, \underset{\sim}{k}, \lambda) \geq w^0(\underset{\sim}{k}, \lambda)$ which implies that the competitive solution achieves the optimum $w^0(\underset{\sim}{k}, \lambda)$. Let $g(\underset{\sim}{k}, \lambda)$ be the maximum utility that an individual can achieve with access to technology $Y(\lambda)$ and initial state $(\underset{\sim}{k}, \lambda)$ but without access to markets. With the additional option to trade at some set of prices, the consumer is not worse off. Hence $v(\underset{\sim}{k}, \underset{\sim}{k}, \lambda) \geq g(\underset{\sim}{k}, \lambda)$. The constant returns to scale technology along with convexity of preferences implies $g(\underset{\sim}{k}, \lambda) = w^0(\underset{\sim}{k}, \lambda)$, establishing the theorem.

7. SUPPORTING PARETO OPTIMUM BY RECURSIVE COMPETITIVE EQUILIBRIUM

Let P be the set of individual value functions for which $v(k, \underset{\sim}{k}, \lambda)$ (i) is continuous and concave in k, (ii) equals $w^0(\underset{\sim}{k}, \lambda)$ if $k = \underset{\sim}{k}$, and (iii) has a zero derivative with respect to $\underset{\sim}{k}$ if $k = \underset{\sim}{k}$. The set P is nonempty since the function $v(k, \underset{\sim}{k}, \lambda) = w^0(\underset{\sim}{k}, \lambda) + w_1^0(\underset{\sim}{k}, \lambda) \cdot (k - \underset{\sim}{k})$ satisfies these conditions. Given $p^* = p^0$ *and* $f^* = \delta_3^0$, the problem facing the individual is a standard concave dynamic programming problem with a unique value function v^* defined below.

Let Tv be the value function which results from the maximization of

$$u(x_1) + \beta \int v(x_3, \delta_3^0(\underset{\sim}{k}, \lambda), \lambda') \, dF(\lambda'|\lambda)$$

subject to $p^0(\underset{\sim}{k}, \lambda) \cdot x \leq 0$ and $x \in X(k)$. As T is a contraction mapping, and maps the set P into itself, the unique fixed point (i.e., bounded function for which $Tv = v$) is an element of P. Let v^* be that element and $x^*(k, \underset{\sim}{k}, \lambda)$ be an optimal policy with the property $x^*(\underset{\sim}{k}, \underset{\sim}{k}, \lambda) = \delta^0(\underset{\sim}{k}, \lambda)$ for all $(\underset{\sim}{k}, \lambda)$. Finally let $y^* = \delta^0$. The set of functions x^*, y^*, p^*, f^*, v^* satisfy the definition of a recursive competitive equilibrium.

COMMENT: This establishes the existence of a recursive competitive equilibrium. The equilibrium will be unique if the equal weight Pareto optimum is unique.

8. EXAMPLES

The model developed by Lucas and Prescott [**13**] to analyze an industry equilibrium with uncertain demand, the asset pricing models of Lucas [**12**] and Brock [**4**], and Sargent's [**18**] analysis of Tobin's "q theory" will be used to illustrate the usefulness of the recursive competitive equilibrium construct developed in this paper.

EXAMPLE 1—*Investment under Uncertainty*: Lucas and Prescott [**13**] consider a competitive industry with constant returns to scale, increasing costs of rapid adjustment, and uncertain demand. The industry is subject to demand shocks λ_t. Letting x_{1t} be industry "output" the downward sloping inverse demand function is

$$D(x_{1t}, \lambda_t). \tag{8.1}$$

The process generating λ_t is a stationary first order Markov process with ergodic set $\Lambda \subseteq R$. The conditional distribution function of λ_{t+1} given λ_t is $F(\lambda_{t+1}|\lambda_t)$. We also assume that the area under the demand curve is uniformly bounded in λ.

Preferences: In order to match the format developed earlier we modify their analysis slightly by introducing identical fictitious individuals. An individual's utility function is:

$$E\left\{\sum_{t=0}^{\infty} \beta^t u(x_{1t}, \lambda_t)\right\}$$

where

$$u(x_1, \lambda) = \int_0^{x_{11}} D(z, \lambda)\, dz + x_{12}. \tag{8.2}$$

Thus the individual supplies the investment good x_{12} infinitely elastically at price 1, and his demand for the consumption good is given by (8.1).

Technology: A constant-returns-to-scale production technology is assumed. The constant-returns-to-scale assumption simplifies the analysis because only total industry capacity or capital is relevant, and not how it is distributed over price-taking firms. Firms face increasing costs of rapid adjustment. A representative firm's production possibility set $Y \subset R^4$ is a closed convex cone constraining the next period's capital y_3, current period output y_{11}, current investment $-y_{12}$, and capital or capacity $-y_2$. The set Y is defined as follows:

$$Y = \{y \in R^4 | 0 \leq y_3 \leq -y_2 g(y_{12}/y_2);\quad 0 \leq y_{11} \leq -y_2;\quad y_{12}, y_2 \leq 0\}$$

where g is strictly increasing and strictly concave and for some $\delta > 0$, $g(\delta) = 1$. An investment of δ times the capital stock is required to maintain the capital stock at its current level and therefore corresponds to a depreciation rate.

In the original analysis the objective of the firm was to maximize the expected present value of its cash flows

$$\sum_{t=0}^{\infty} \beta^t\{p_{11t}y_{11t}+y_{12t}\}, \quad 0<\beta<1,$$

subject to $y_t \in Y$ for all t. For our structure the firm need only solve a static maximization problem each period—namely, maximize $p_t \cdot y_t$. This is accomplished by adding the capital stock used this period and capital stock available for production next period to the commodity vector y_t.

The consumers consumption possibility set is

$$X(k)=\{x \in R^4|x_{11}\geqslant 0;\ x_{12}\leqslant 0;\ -x_2\leqslant k;\ x_3\geqslant 0\}.$$

All the assumptions of the recursive competitive equilibrium theory are satisfied and hence we can establish the existence of an industry equilibrium. (See Lucas and Prescott [**13**, Theorem 1].) Further, the industry equilibrium investment function depends on $\underline{k}$ and λ and maximizes discounted expected utility.

EXAMPLE 2—*Lucas' Capital Asset Pricing Model*: The production technology for Lucas' [**12**] model is very simple. Output of firm j is constrained by λ_j for $j=1,2,\ldots n$. The n-component vector λ of the λ_j is subject to a stationary first order Markov process. $F(\lambda'|\lambda)$ denotes the distribution function of next period's constraint vector, λ', given the current constraint vector λ.

It is necessary to consider a slightly more general structure in order to price the shares of the firms. We assume that output of the consumption good $y_{1j}\in R$ is constrained as follows:

$$0\leqslant y_{1j}\leqslant -y_{2jj}\lambda_j$$

where $-y_{2jj}$ is the "number of rights" to use process j purchased by price-taking firm j. The production of rights to use process j next period by firm j, y_{3jj}, is constrained as follows:

$$0\leqslant y_{3jj}\leqslant -y_{2jj},$$

so production rights cannot be increased. The firm chooses $y_j=(y_{1j}, y_{2j}, y_{3j})\in R^{2n+1}$ subject to the above constraints, which define the production possibility set $Y(\lambda)$, so as to maximize period valuation. The consumer's utility function is

$$\sum_{t=0}^{\infty} \beta^t u(x_{1t})$$

and the consumption possibility set is

$$X(k)=\{x\in R^{2n+1}|x_1\geqslant 0; \quad 0\geqslant x_2\geqslant -k; \quad x_3\geqslant 0\}.$$

The initial value of k is a vector of ones; so that k corresponds to shares.

All the assumptions for the existence of a recursive competitive equilibrium are satisfied. Further,

$$v(\underline{k}, \underline{k}, \lambda) = u\left(\sum_{j=1}^{n} \lambda_j \underline{k}_j\right) + \beta \int v(\underline{k}, \underline{k}, \lambda')\, dF(\lambda'|\lambda)$$

as the optimal x_1 is to consume all of the perishable goods that can be produced. The equilibrium of interest is the one for which $\underline{k}_j = 1$ for all j for then $\underline{k}_j$ corresponds to shares. The first-order conditions associated with the consumer's maximization problem give

$$p_{3jt} = p_{2jt} - \lambda_{jt} = \beta E_t\{[u'(x_{1,t+1})/u'(x_{1t})]p_{2j,t+1}\}.$$

This result is the same as equation 6 in Lucas [**12**].

EXAMPLE 3—*Brock's Capital Asset Pricing Model*: Brock [**4**], using non-recursive methods, analyzes the pricing of capital assets using a general equilibrium model. In his model the structure of preference and technology is recursive and can be mapped into the structure considered here as follows:

Technology: The contribution of process j to total output available this period for allocation to current consumption and investment is a function of the capital of type j allocated to the process, s_j, and random shock λ. Output of the jth process is $f_j(s_j, \lambda)$ where f_j is increasing and concave in s_j and uniformly bounded over all $(s, \lambda) \in [0, \infty] \times \Lambda$. Brock also considers the case for which the production functions are not necessarily bounded but requires a bounded utility function. Our analysis can be modified to analyze this case. We do not do so because the resulting equilibrium process will typically not have a stationary distribution making it of less econometric interest.

The finite dimension vector of random shocks is identically and independently distributed over time. This production technology does not display constant returns to scale which necessitates the introduction of another factor denoting the right to use process j. Output is constrained by $z_j f_j(s_j/z_j, \lambda)$ where z_j is positive. This function is homogeneous of degree one in the inputs and outputs and for $z_j = 1$ places the appropriate constraint on current output. Letting c be consumption, s_j' investment in capital of type j (next period's capital of type j), and z_j' the number of next period rights, the production possibility set is:

$$Y(\lambda) = \Big\{(c, -z, -s, z', s') \in R^{4J+1} | c, z, s, z', s' \geq 0;$$
$$c + \sum_{j \in J} s_j' \leq \sum_{j \in J} z_j f_j(s_j/z_j, \lambda); \quad z_j' \leq z_j\Big\}$$

where $z = (z_1, \ldots, z_j)$ and $s = (s_1, \ldots, s_j)$. The z_j correspond to rights to use process j and the initial endowment owned by the consumer is one. In equilibrium $z_{jt} = 1$ for all j and t. The price of z_j is the market value of the firm prior to the distribution of dividends. The dividends paid by firm j (possibly negative, if the firm invests more than it produces that period) is equal to the price of z_j less the price of z_j'. The price of z_j' is the ex dividend stock price while z_j is the pre dividend value.

Preferences: The commodity point $(c, -z, -s, z', s')$ is an element of R^{4J+1}. Relative to our notation, c corresponds to x_1, $(-z, -s)$ to x_2, and (z', s') to x_3.

The capital k owned by the consumer at the beginning of the period constrains the factors supplied and is an element in the positive orthant of R^{2J}. The consumption possibility set is

$$X(k) = \{(c, -z, -s, z', s') \in R^{4J+1} | c, z, s, z', s' \geq 0; \quad (z, s) \leq k\}.$$

The consumer's utility function, whose expected value is maximized, is

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

where function u satisfies all the usual assumptions. The initial holding of rights to use process j is one for all $j = 1, \ldots, J$.

The assumptions of our recursive equilibrium theory are again satisfied. Consequently a recursive competitive equilibrium exists and is optimal. With strict concavity of the utility and production functions, the equilibrium allocation is unique. If, in addition, technology and references are sufficiently smooth to insure a unique plane separating the production set $Y(\lambda)$ and the set of commodity points $\underline{x}$ for which

$$u(\underline{x}_1) + \beta \int v(\underline{x}_3, \underline{x}_3, \lambda') \, dF(\lambda'|\lambda) \geq v(\underline{k}, \underline{k}, \lambda)$$

is unique for all $\underline{x} \in X(\underline{k})$, then the equilibrium pricing function is unique as well.

EXAMPLE 4—*Sargent's Analysis of Tobin's "q theory"*: Sargent **[18]** analyzes Tobin's "q theory" in a general equilibrium framework using techniques related to those developed here. This model can easily be analyzed within our framework. Sargent uses a putty-clay version of the stochastic one-sector growth model of Mirman and Zilcha **[16]** as a vehicle for making some observations about the "q theory" of investment. There is one capital good "putty" that can either be consumed or be rented to firms at a competitively determined rate. However, once in place capital cannot be consumed (hence "putty-clay"). The individuals also supply one unit of labor at a competitively determined wage rate w. Firms use capital and labor to produce "putty" and either consume it or rent it out and so it goes on.

Technology: Constant returns to scale technology is assumed. The production function relating output of new "putty" to input labor n and capital k is $nf(k/n)\theta$. Sargent considers the case when the individual inelastically supplies one unit of labor, i.e., $n = 1$ and assumes $f'(\cdot) > 0$, $f'(0) = \infty$, and $f'(\infty) = 0$. Shock θ is identically and independently distributed over time.

Preferences: The representative individual maximizes $E\{\sum_{t=0}^{\infty} \beta^t u(c_t, \varepsilon_t)\}$, $0 < \beta < 1$, where u is bounded, differentiable, increasing, and concave in c. Shocks ε_t are independently and identically distributed over time and are independent of θ.

The period commodity point is $(c, -n, x_2, x_3) \in R^4$. The vector $(c, -n)$ corresponds to x_1; x_2 is capital supplied and is constrained by available capital k; x_3 is next period's capital. The constraints on the consumption possibility set $X(k)$ are $X(k) = \{(c, -n, x_2, x_3) \in R^4 | c \geqslant 0;\ 0 \leqslant n \leqslant 1;\ x_2 \geqslant -k;\ x_3 \geqslant 0\}$. The production possibility set $Y(\lambda)$, where $\lambda = (\theta, \varepsilon)$, is

$$Y(\lambda) = \{(c, -n, x_2, x_3) \in R^4 | c, n, -x_2, x_3 \geqslant 0;$$
$$c + x_3 \leqslant nf(-x_2/n)\theta - (1-\delta)x_2;\quad c \leqslant nf(-x_2/n)\theta\}.$$

The set of obtainable commodity points is bounded since $f'(\infty) = 0$ and depreciation rate δ is positive. All the assumptions for the recursive competitive equilibrium are satisfied and consequently the equilibrium exists.

The above examples illustrate that the assumptions of the recursive competitive equilibrium theory are satisfied for an interesting class of infinite horizon equilibrium problems.

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